

# The derivative of sine and cosine

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## Outline: Main goals

Evaluate two important trigonometric limits:

- ▶ Evaluate  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$
- ▶ Evaluate  $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$

## Outline: Main goals

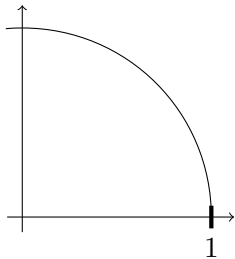
Evaluate two important trigonometric limits:

- ▶ Evaluate  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$
- ▶ Evaluate  $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$

Use these to:

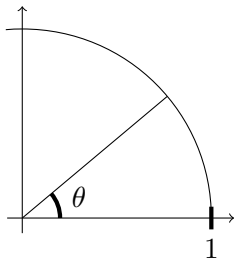
- ▶ Find the derivative of  $f(x) = \sin x$
- ▶ Find the derivative of  $f(x) = \cos x$

## Some trigonometry/geometry



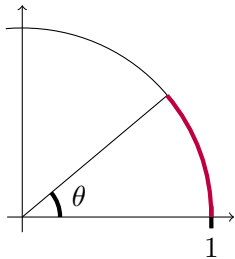
Choose any (small) angle  $\theta$

## Some trigonometry/geometry



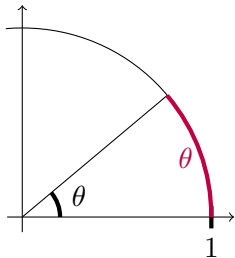
Choose any (small) angle  $\theta$

## Some trigonometry/geometry



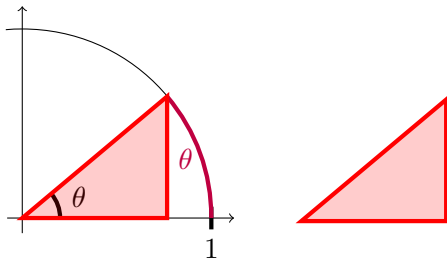
Highlight arc in purple

## Some trigonometry/geometry



Arclength formula  $s = r\theta$ : length  $\theta$

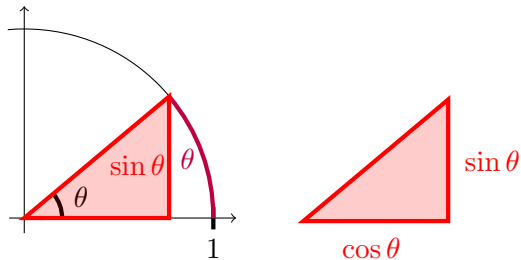
## Some trigonometry/geometry



triangle of hypotenuse 1... a “unit circle” triangle

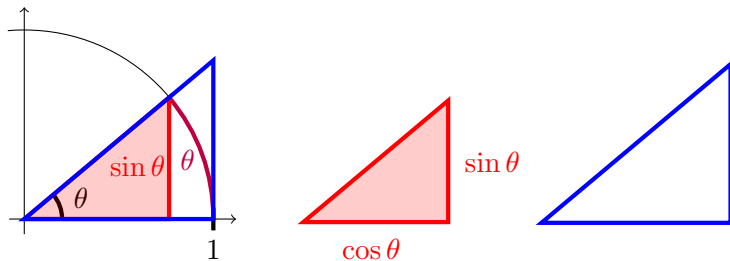


## Some trigonometry/geometry



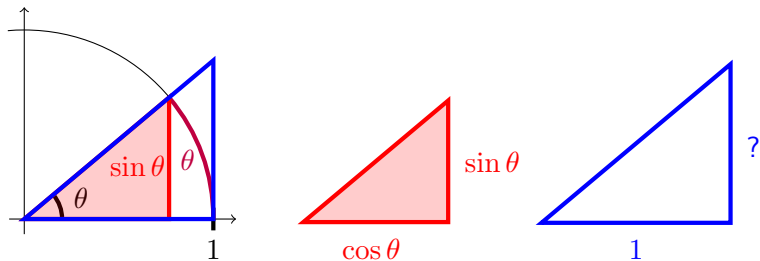
triangle of hypotenuse 1... a “unit circle” triangle

## Some trigonometry/geometry



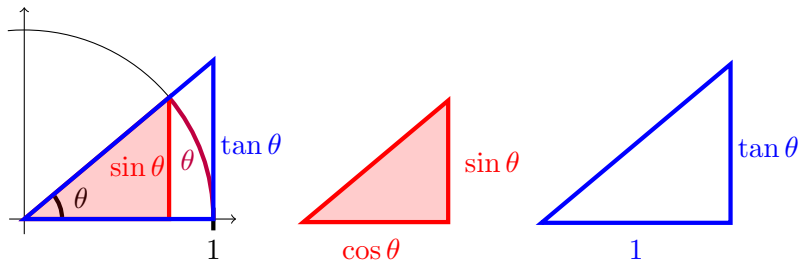
bigger (similar) triangle (in blue)

## Some trigonometry/geometry



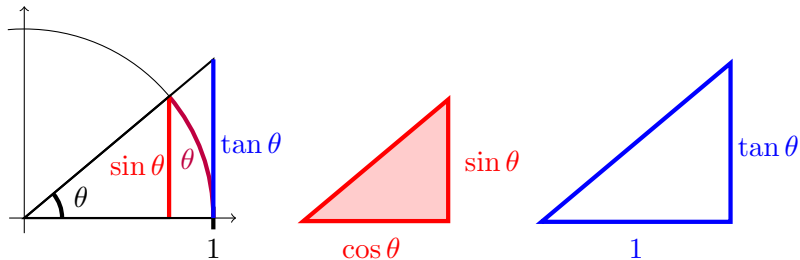
what is the missing side? use similar triangles:  $\frac{\sin \theta}{\cos \theta} = \frac{?}{1}$

## Some trigonometry/geometry

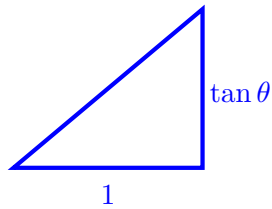
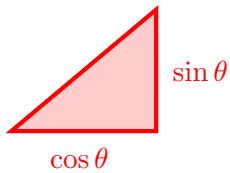
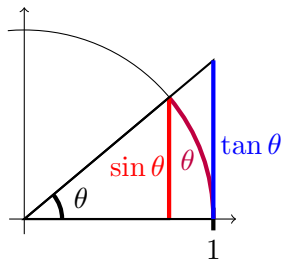


what is the missing side? use similar triangles:  $\frac{\sin \theta}{\cos \theta} = \frac{?}{1}$

## Some trigonometry/geometry

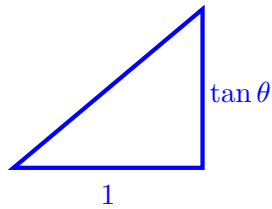
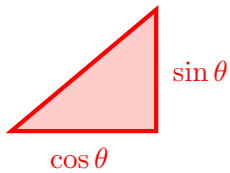
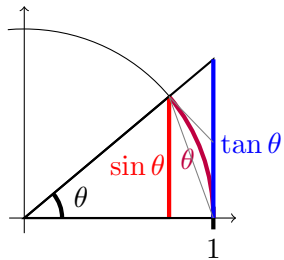


## Some trigonometry/geometry



$$\tan \theta \geq \theta \geq \sin \theta$$

## Some trigonometry/geometry



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$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

Multiply by  $\sin \theta$

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$$

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

Multiply by  $\sin \theta$

$$\underbrace{\cos \theta}_{f(\theta)} \leq \underbrace{\frac{\sin \theta}{\theta}}_{h(\theta)} \leq \underbrace{1}_{h(\theta)}$$

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

Multiply by  $\sin \theta$

$$\underbrace{\cos \theta}_{f(\theta)} \leq \underbrace{\frac{\sin \theta}{\theta}}_{g(\theta)} \leq \underbrace{1}_{h(\theta)}$$

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

Multiply by  $\sin \theta$

$$\underbrace{\cos \theta}_{f(\theta)} \leq \underbrace{\frac{\sin \theta}{\theta}}_{g(\theta)} \leq \underbrace{1}_{h(\theta)}$$

►  $\lim_{\theta \rightarrow 0} f(\theta) = 1$  and  $\lim_{\theta \rightarrow 0} h(\theta) = 1$

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

Multiply by  $\sin \theta$

$$\underbrace{\cos \theta}_{f(\theta)} \leq \underbrace{\frac{\sin \theta}{\theta}}_{g(\theta)} \leq \underbrace{1}_{h(\theta)}$$

- ▶  $\lim_{\theta \rightarrow 0} f(\theta) = 1$  and  $\lim_{\theta \rightarrow 0} h(\theta) = 1$
- ▶ By the Squeeze Theorem,  $\lim_{\theta \rightarrow 0} g(\theta) = 1$ .

$$\tan \theta \geq \theta \geq \sin \theta$$

$$\frac{\sin \theta}{\cos \theta} \geq \theta \geq \sin \theta$$

Take the reciprocal:

$$\frac{\cos \theta}{\sin \theta} \leq \frac{1}{\theta} \leq \frac{1}{\sin \theta}$$

Multiply by  $\sin \theta$

$$\underbrace{\cos \theta}_{f(\theta)} \leq \underbrace{\frac{\sin \theta}{\theta}}_{g(\theta)} \leq \underbrace{1}_{h(\theta)}$$

- ▶  $\lim_{\theta \rightarrow 0} f(\theta) = 1$  and  $\lim_{\theta \rightarrow 0} h(\theta) = 1$
- ▶ By the Squeeze Theorem,  $\lim_{\theta \rightarrow 0} g(\theta) = 1$ .
- ▶ In other words,  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ .