# The product rule for differentation

E. Kim

# Product Rule for Differentiation

### Goal

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

# Product Rule for Differentiation

**Goal**

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

Previously, we saw $[f(x)+g(x)]' = f'(x) + g'(x)$ **"Sum Rule"**

# Product Rule for Differentiation

## Goal

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

Previously, we saw $[f(x)+g(x)]' = f'(x)+g'(x)$ **"Sum Rule"**

## Question

Is the Product Rule

$$[f(x)\,g(x)]' = f'(x)\,g'(x)$$

or not?

Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

# Is $[f(x)\, g(x)]' = f'(x)\, g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad\qquad g(x) = x^{10}$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$f(x) = x^3$ $\qquad$ $g(x) = x^{10}$ $\qquad$ $k(x) = f(x) \cdot g(x) = x^{13}$

Compute derivatives:

$f'(x) =$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$f(x) = x^3$ $\qquad$ $g(x) = x^{10}$ $\qquad$ $k(x) = f(x) \cdot g(x) = x^{13}$

Compute derivatives:

$f'(x) = 3x^2$ $\qquad$ $g'(x) =$

## Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9 \qquad k'(x) = [f(x)\,g(x)]' =$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9 \qquad k'(x) = [f(x)\,g(x)]' = 13x^{12}$$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9 \qquad k'(x) = [f(x)\,g(x)]' = 13x^{12}$$

Compare:

## Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9 \qquad k'(x) = [f(x)\,g(x)]' = 13x^{12}$$

Compare:

- $[f(x)\,g(x)]' = 13x^{12}$

# Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9 \qquad k'(x) = [f(x)\,g(x)]' = 13x^{12}$$

Compare:

- $[f(x)\,g(x)]' = 13x^{12}$
- $f'(x)\,g'(x) = (3x^2)(10x^9) = 30x^{11}$

## Is $[f(x)\,g(x)]' = f'(x)\,g'(x)$ the Product Rule?

Let's test it out! Choose:

$$f(x) = x^3 \qquad g(x) = x^{10} \qquad k(x) = f(x) \cdot g(x) = x^{13}$$

Compute derivatives:

$$f'(x) = 3x^2 \qquad g'(x) = 10x^9 \qquad k'(x) = [f(x)\,g(x)]' = 13x^{12}$$

Compare:

- $[f(x)\,g(x)]' = 13x^{12}$
- $f'(x)\,g'(x) = (3x^2)(10x^9) = 30x^{11}$

**No! This is NOT the Product Rule!**

$[f(x)\,g(x)]' \neq f'(x)\,g'(x)$

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta r$

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta r$
- Hours per week goes up: $h \rightsquigarrow h_{\text{new}}$
  $h_{\text{new}} = h + \Delta h$

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta r$
- Hours per week goes up: $h \rightsquigarrow h_{\text{new}}$
  $h_{\text{new}} = h + \Delta h$
- Both $r$ and $h$ increase

# Then what **is** the Product Rule?

## Intuitively… it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

- $p_{\text{old}} = rh$

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta r$
- Hours per week goes up: $h \rightsquigarrow h_{\text{new}}$
  $h_{\text{new}} = h + \Delta h$
- Both $r$ and $h$ increase

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta\, r$
- Hours per week goes up: $h \rightsquigarrow h_{\text{new}}$
  $h_{\text{new}} = h + \Delta\, h$
- Both $r$ and $h$ increase

- $p_{\text{old}} = rh$
- $p_{\text{new}} = r_{\text{new}}\, h_{\text{new}} = (r + \Delta r)(h + \Delta h)$

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta\, r$
- Hours per week goes up: $h \rightsquigarrow h_{\text{new}}$
  $h_{\text{new}} = h + \Delta\, h$
- Both $r$ and $h$ increase

- $p_{\text{old}} = rh$
- $p_{\text{new}} = r_{\text{new}}\, h_{\text{new}} = (r + \Delta r)(h + \Delta h)$
- Change in pay $\Delta p = p_{\text{new}} - p_{\text{old}} = (r + \Delta r)(h + \Delta h) - rh$

# Then what **is** the Product Rule?

## Intuitively... it's like working at Culver's

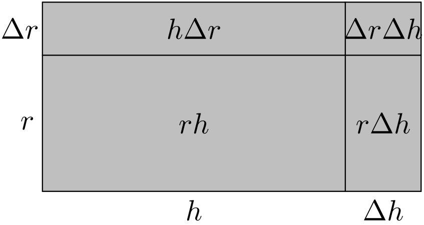
Culver's in Onalaska, WI
*Source: Wikipedia*

Say you worked at Culver's at a rate of $r = 7.75$ per hour for $h = 20$ hours each week. Your take-home pay is $p = rh$. How can your take-home pay go up?

- Pay rate goes up: $r \rightsquigarrow r_{\text{new}}$
  $r_{\text{new}} = r + \Delta\, r$
- Hours per week goes up: $h \rightsquigarrow h_{\text{new}}$
  $h_{\text{new}} = h + \Delta\, h$
- Both $r$ and $h$ increase

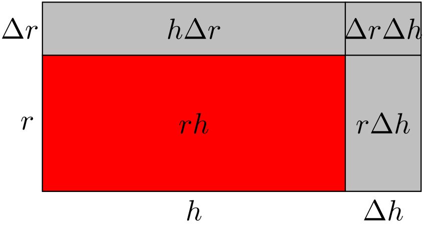
- $p_{\text{old}} = rh$
- $p_{\text{new}} = r_{\text{new}}\, h_{\text{new}} = (r + \Delta r)(h + \Delta h)$
- Change in pay $\Delta p = p_{\text{new}} - p_{\text{old}} = (r + \Delta r)(h + \Delta h) - rh = (rh + r\,\Delta h + h\,\Delta r + \Delta r\,\Delta h) - rh$

# Intuitive idea of the Product Rule



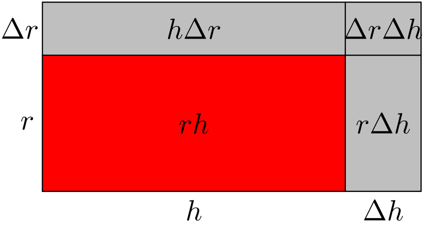

$$\Delta p = (rh + r\,\Delta h + h\,\Delta r + \Delta r\,\Delta h) - rh$$

# Intuitive idea of the Product Rule




$$\Delta p = (rh + r\,\Delta h + h\,\Delta r + \Delta r\,\Delta h) - rh$$

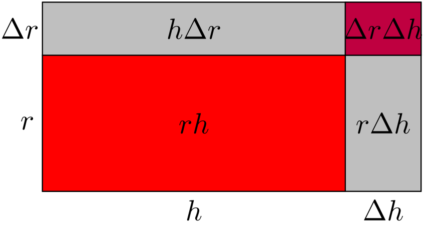
# Intuitive idea of the Product Rule




$$\Delta p = (rh + r\,\Delta h + h\,\Delta r + \Delta r\,\Delta h) - rh$$

$$\Delta p = r\,\Delta h + h\,\Delta r + \underbrace{\Delta r\,\Delta h}$$

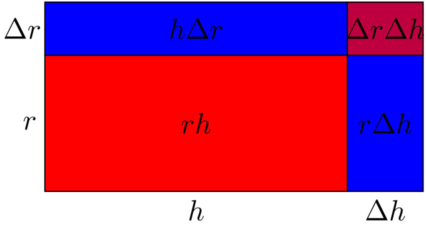
# Intuitive idea of the Product Rule




$$\Delta p = (rh + r\,\Delta h + h\,\Delta r + \Delta r\,\Delta h) - rh$$

$$\Delta p = r\,\Delta h + h\,\Delta r + \underbrace{\Delta r\,\Delta h}_{\text{negligible}}$$

# Intuitive idea of the Product Rule




$$\Delta p = (rh + r\,\Delta h + h\,\Delta r + \Delta r\,\Delta h) - rh$$

$$\Delta p = r\,\Delta h + h\,\Delta r + \underbrace{\Delta r\,\Delta h}_{\text{negligible}}$$

$$\Delta p \approx r\,\Delta h + h\,\Delta r$$

The change in the product $p = rh$ is the old rate $r$ times the change in hours $(\Delta h)$, plus the old hours $h$ times the change in rate $(\Delta r)$.

# Deriving the Product Rule

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

# Deriving the Product Rule

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

By definition of derivative,

$$[f(x)\,g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

# Deriving the Product Rule

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

By definition of derivative,

$$[f(x)\,g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Subtract and add $f(x+h)\,g(x)$ in the numerator:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)g(x)}{h}$$

# Deriving the Product Rule

Starting with differentiable functions $f(x)$ and $g(x)$, we want to get the derivative of $f(x)\,g(x)$.

By definition of derivative,

$$[f(x)\,g(x)]' = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Subtract and add $f(x+h)\,g(x)$ in the numerator:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)g(x)}{h}$$

Sum law for limits

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)\,g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)\,g(x)-f(x)g(x)}{h}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)\,g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)\,g(x)-f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0}\frac{f(x+h)\big(g(x+h)-g(x)\big)}{h}+\lim_{h\to 0}\frac{g(x)\big(f(x+h)-f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}\underbrace{\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)}+\underbrace{\left(\lim_{h\to 0} g(x)\right)}\underbrace{\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)\,g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)\,g(x)-f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0}\frac{f(x+h)\big(g(x+h)-g(x)\big)}{h}+\lim_{h\to 0}\frac{g(x)\big(f(x+h)-f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{f\text{ is diff.,}}\underbrace{\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)}+\underbrace{\left(\lim_{h\to 0} g(x)\right)}\underbrace{\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{\substack{f \text{ is diff.,} \\ \text{so } f \text{ is cts}}} \underbrace{\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right)} + \underbrace{\left(\lim_{h\to 0} g(x)\right)} \underbrace{\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{\substack{f \text{ is diff.,} \\ \text{so } f \text{ is cts} \\ \text{so this } = f(x)}} \underbrace{\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right)} + \underbrace{\left(\lim_{h\to 0} g(x)\right)} \underbrace{\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{\substack{f \text{ is diff.,} \\ \text{so } f \text{ is cts} \\ \text{so this } = f(x)}} \underbrace{\left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)} + \underbrace{\left(\lim_{h\to 0} g(x)\right)}_{\substack{= g(x)\text{, since} \\ \text{does not} \\ \text{depend on } h}} \underbrace{\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{\substack{f \text{ is diff.,} \\ \text{so } f \text{ is cts} \\ \text{so this } = f(x)}} \underbrace{\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right)}_{g'(x),\text{ by defn. of deriv.}} + \underbrace{\left(\lim_{h\to 0} g(x)\right)}_{\substack{=g(x),\text{ since} \\ \text{does not} \\ \text{depend on } h}} \underbrace{\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)}_{f'(x),\text{ by defn. of deriv.}}$$

# Continued...

From previous slide, $[f(x)\,g(x)]'$ is equal to:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)\,g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)g(x)}{h}$$

Factor:

$$\lim_{h\to 0} \frac{f(x+h)\big(g(x+h) - g(x)\big)}{h} + \lim_{h\to 0} \frac{g(x)\big(f(x+h) - f(x)\big)}{h}$$

Product law for limits

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{\substack{f \text{ is diff.,}\\ \text{so } f \text{ is cts}\\ \text{so this } = f(x)}} \underbrace{\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right)}_{g'(x)\text{, by defn. of deriv.}} + \underbrace{\left(\lim_{h\to 0} g(x)\right)}_{\substack{= g(x)\text{, since}\\ \text{does not}\\ \text{depend on } h}} \underbrace{\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)}_{f'(x)\text{, by defn. of deriv.}}$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

# The Product Law for Derivatives

If $f = f(x)$ and $g = g(x)$ are differentiable, the derivative of the product is given by:

## The Product Law: in Newton notation

$$[f(x)g(x)]' = f(x)\, g'(x) + g(x)\, f'(x)$$

## The Product Law: in Leibniz notation

$$\frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$$

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\, g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

Using the Product Rule...

**...in Newton notation**

$$[f(x)g(x)]'$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

**...in Leibniz notation**

$$\frac{d}{dx}[fg]$$

$$f\frac{dg}{dx} + g\frac{df}{dx}$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

Using the Product Rule...

## ...in Newton notation

$$[f(x)g(x)]'$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

## ...in Leibniz notation

$$\frac{d}{dx}[fg]$$

$$f\frac{dg}{dx} + g\frac{df}{dx}$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

Using the Product Rule...

**...in Newton notation**

$$[f(x)g(x)]'$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

**...in Leibniz notation**

$$\frac{d}{dx}[fg]$$

$$f\frac{dg}{dx} + g\frac{df}{dx}$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

Using the Product Rule...

**...in Newton notation**

$$[f(x)g(x)]'$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

**...in Leibniz notation**

$$\frac{d}{dx}[fg]$$

$$f\frac{dg}{dx} + g\frac{df}{dx}$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

Using the Product Rule...

### ...in Newton notation

$$[f(x)g(x)]'$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

### ...in Leibniz notation

$$\frac{d}{dx}[fg]$$

$$f\frac{dg}{dx} + g\frac{df}{dx}$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

# Revisit earlier example

Earlier, had $f(x) = x^3$, $g(x) = x^{10}$, and $k(x) = f(x)\,g(x) = x^{13}$.
The derivative of the product was $k'(x) = 13x^{12}$.

Using the Product Rule...

**...in Newton notation**

$$[f(x)g(x)]'$$

$$f(x)\,g'(x) + g(x)\,f'(x)$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

**...in Leibniz notation**

$$\frac{d}{dx}[fg]$$

$$f\frac{dg}{dx} + g\frac{df}{dx}$$

$$(x^3)(10x^9) + (x^{10})(3x^2)$$

in either notation,

$$10x^{12} + 3x^{12}$$

$$= 13x^{12}$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\, g(x)$, where

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\, g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$k'(x) = f(x)\,g'(x) + g(x)\,f'(x)$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\, g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$\begin{aligned} k'(x) &= f(x)\, g'(x) + g(x)\, f'(x) \\ &= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x) \end{aligned}$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x) \end{aligned}$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x) \end{aligned}$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x) \end{aligned}$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\, g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$
\begin{aligned}
k'(x) &= f(x)\, g'(x) + g(x)\, f'(x) \\
&= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x)
\end{aligned}
$$

# Example

**Exercise:** Let $k(x) = (x^2+1)(x^3+5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2+1$ and $g(x) = x^3+5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2+5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2+1)(3x^2+5) + (x^3+5x)(2x) \\ &= 5x^4 + 18x^2 + 5 \end{aligned}$$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x) \\ &= 5x^4 + 18x^2 + 5 \end{aligned}$$

**Solution 2:**

# Example

**Exercise:** Let $k(x) = (x^2+1)(x^3+5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2+1$ and $g(x) = x^3+5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2+5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2+1)(3x^2+5) + (x^3+5x)(2x) \\ &= 5x^4 + 18x^2 + 5 \end{aligned}$$

**Solution 2:**

- FOIL out $k(x)$ to get $k(x) = x^5 + 6x^3 + 5x$

# Example

**Exercise:** Let $k(x) = (x^2 + 1)(x^3 + 5x)$. Differentiate $k(x)$.
**Solution:**

- $k(x) = f(x)\,g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x^3 + 5x$.
- $f'(x) = 2x$ and $g'(x) = 3x^2 + 5$.

$$\begin{aligned} k'(x) &= f(x)\,g'(x) + g(x)\,f'(x) \\ &= (x^2 + 1)(3x^2 + 5) + (x^3 + 5x)(2x) \\ &= 5x^4 + 18x^2 + 5 \end{aligned}$$

**Solution 2:**

- FOIL out $k(x)$ to get $k(x) = x^5 + 6x^3 + 5x$
- Sum Rule and Power Rule: $k'(x) = 5x^4 + 18x^2 + 5$

# Example [B]

**Exercise:** Find $\frac{d}{dx}\left[\frac{1}{x}(x^2+e^x)\right]$.

# Example [B]

**Exercise:** Find $\frac{d}{dx}\left[\frac{1}{x}(x^2+e^x)\right]$.

**Solution:**

- $f(x) = \frac{1}{x}$ and $g(x) = x^2 + e^x$

$$\begin{aligned}
\frac{d}{dx}\left[f(x)\,g(x)\right] &= f\frac{dg}{dx} + g\frac{df}{dx} \\
&= \frac{1}{x}\frac{d}{dx}\left[x^2+e^x\right] + (x^2+e^x)\frac{d}{dx}\left[\frac{1}{x}\right] \\
&= \frac{1}{x}\frac{d}{dx}\left[x^2+e^x\right] + (x^2+e^x)\frac{d}{dx}\left[x^{-1}\right] \\
&= \frac{1}{x}(2x+e^x) + (x^2+e^x)(-1x^{-2})
\end{aligned}$$

# Example [B]

**Exercise:** If $y = x^3 e^x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

In other words, find $y'$ and $y''$.

# Example [B]

**Exercise:** If $y = x^3 e^x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

In other words, find $y'$ and $y''$.

**Solution:**

- $y' = (x^3)(e^x)' + (x^3)'(e^x) = x^3e^x + 3x^2e^x$
- $y'' = [x^3e^x + (3x^2)(e^x)]' =$
  $(x^3e^x + 3x^2e^x) + (3x^2)(e^x)' + (3x^2)'(e^x)$
- $y'' = x^3e^x + 3x^2e^x + 3x^2e^x + 6xe^x = x^3e^x + 6x^2e^x + 6xe^x$

# Example [B]

**Exercise:** What is the derivative of $f(x) = \sqrt{x}(3x + 2)$?

# Example [B]

**Exercise:** What is the derivative of $f(x) = \sqrt{x}(3x + 2)$?

**Solution:**
$f'(x) = (\sqrt{x})(3x + 2)' + (\sqrt{x})'(3x + 2) = \sqrt{x}(3) + \frac{1}{2\sqrt{x}}(3x + 2)$

$$= 3\sqrt{x} + \frac{3x + 2}{2\sqrt{x}}$$

# Example [B]

**Exercise:** What is the derivative of $f(x) = (x + 1)(x^2 - 7x)(e^x)$?

# Example [B]

**Exercise:** What is the derivative of $f(x) = (x+1)(x^2-7x)(e^x)$?

**Solution:**

- Think of $f$ as being: $[(x+1)(x^2-7x)][e^x]$

$$f' = [(x+1)(x^2-7x)]'[e^x] + [(x+1)(x^2-7x)][e^x]'$$
$$= \Big[(x+1)(x^2-7x)' + (x+1)'(x^2-7x)\Big][e^x] + [(x+1)(x^2-7x)][e^x]'$$
$$= \Big[(x+1)(2x-7) + (1)(x^2-7x)\Big][e^x] + [(x+1)(x^2-7x)][e^x]$$

- First break the function of **three** factors into two factors: a "super factor" and a regular factor, then use the Product Rule twice
- Or, use the "Triple Product Rule", proved by doing Product Rule twice on a generic "super factor"

# Example

**Exercise:** If $k(x) = f(x) \cdot g(x)$ and

- $f(2) = 3$
- $f'(2) = -4$
- $g(2) = 1$
- $g'(2) = 5$

then find $k'(2)$.

# Example

**Exercise:** If $k(x) = f(x) \cdot g(x)$ and

- $f(2) = 3$
- $f'(2) = -4$
- $g(2) = 1$
- $g'(2) = 5$

then find $k'(2)$.

**Solution:**

# Example

**Exercise:** If $k(x) = f(x) \cdot g(x)$ and

- $f(2) = 3$
- $f'(2) = -4$
- $g(2) = 1$
- $g'(2) = 5$

then find $k'(2)$.

**Solution:**

- Use $k'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, plug in $x = 2$

# Example

**Exercise:** If $k(x) = f(x) \cdot g(x)$ and

- $f(2) = 3$
- $f'(2) = -4$
- $g(2) = 1$
- $g'(2) = 5$

then find $k'(2)$.

**Solution:**

- Use $k'(x) = f(x)\, g'(x) + g(x)\, f'(x)$, plug in $x = 2$

$$k'(2) = f(2)\, g'(2) + g(2)\, f'(2)$$

# Example

**Exercise:** If $k(x) = f(x) \cdot g(x)$ and

- $f(2) = 3$
- $f'(2) = -4$
- $g(2) = 1$
- $g'(2) = 5$

then find $k'(2)$.

**Solution:**

- Use $k'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, plug in $x = 2$

$$\begin{aligned} k'(2) &= f(2)\,g'(2) + g(2)\,f'(2) \\ &= (3)(5) + (1)(-4) \end{aligned}$$

# Example

**Exercise:** If $k(x) = f(x) \cdot g(x)$ and

- $f(2) = 3$
- $f'(2) = -4$
- $g(2) = 1$
- $g'(2) = 5$

then find $k'(2)$.

**Solution:**

- Use $k'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, plug in $x = 2$

$$\begin{aligned} k'(2) &= f(2)\,g'(2) + g(2)\,f'(2) \\ &= (3)(5) + (1)(-4) \\ &= 11 \end{aligned}$$

# Example [B]

**Exercise:** Differentiate $f(x) = (10)(x^6)$

# Example [B]

**Exercise:** Differentiate $f(x) = (10)(x^6)$

**Solution:** $\dfrac{df}{dx} = \dfrac{d}{dx}[10] \cdot (x^6) + (10)\dfrac{d}{dx} \cdot [x^6]$
which simplifies $0 \cdot (x^6) + (10)6x^5 = 60x^5$

# Example [B]

**Exercise:** Differentiate $f(x) = (10)(x^6)$

**Solution:** $\frac{df}{dx} = \frac{d}{dx}[10] \cdot (x^6) + (10)\frac{d}{dx} \cdot [x^6]$
which simplifies $0 \cdot (x^6) + (10)6x^5 = 60x^5$

**FASTER SOLUTION:** $\frac{d}{dx}\left[10x^6\right] = 10\frac{d}{dx}\left[x^6\right] = 10 \cdot 6x^5 = 60x^5$

# Example [B]

**Exercise:** Differentiate $f(x) = (10)(x^6)$

**Solution:** $\dfrac{df}{dx} = \dfrac{d}{dx}[10] \cdot (x^6) + (10)\dfrac{d}{dx} \cdot [x^6]$
which simplifies $0 \cdot (x^6) + (10)6x^5 = 60x^5$

**FASTER SOLUTION:** $\dfrac{d}{dx}\left[10x^6\right] = 10\dfrac{d}{dx}\left[x^6\right] = 10 \cdot 6x^5 = 60x^5$

**Time-saving tip!**

Just because you **can** use the Product Rule doesn't mean that you always **should**.

- If one of your factors is just a **constant**, then **SAVE SOME TIME** by using the Constant Multiple Rule instead!!!