16.4: Green's Theorem in the Plane[1]

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The setup is:

- $M(x,y)$ and $N(x,y)$ are scalar functions with continuous first partial derivatives
- C is a piecewise smooth, simple closed curve enclosing a region R in the xy-plane

Then,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy \tag{1}$$

and

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy \tag{2}$$

If, in addition $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$, then:

- Comments on equation (1) are:
  - The left part of equation (1) is called the circulation of $\mathbf{F}$ around C and is also written
    $$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds. \tag{3}$$
  - The expression in parentheses in equation (1) is called the circulation density of $\mathbf{F}$. It is also called the k-component of curl of $\mathbf{F}$. In symbols:
    $$(\operatorname{curl}\mathbf{F})\cdot\mathbf{k} \text{ is defined to be } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$
  - For this reason, equation (1) is called the circulation-curl version of Green's Theorem[2].
- Comments on equation (2) are:
  - The left part of equation (2) is called the flux of $\mathbf{F}$ across C and is also written
    $$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds. \tag{4}$$
  - The expression in parentheses in equation (2) is called the flux density of $\mathbf{F}$. It is also called the divergence of $\mathbf{F}$. In symbols:
    $$(\operatorname{div}\mathbf{F}) \text{ is defined to be } \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$
  - For this reason, equation (2) is called the flux-divergence version of Green's Theorem[3].

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[1] Notation follows Notation follows Thomas' Calculus: Early Transcendentals (12th Edition) as closely as possible

[2] Also called the Tangential Form of Green's Theorem because of the **T** in formula (3).

[3] Also called the Normal Form of Green's Theorem because of the **n** in formula (4).