How to compute line integrals of vector fields

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Physics: Work

In physics, there's a formula that says

$$\text{Work} = \text{Force} \times \text{Distance}$$

If $\mathbf{F}$ is a force field (whether gravitational or otherwise), then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the work done by the force in moving an object along the path C parametrized as $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \le t \le b$

How to compute line integrals of vector fields

Given a vector field $\mathbf{F}(x, y, z) = \underbrace{M(x, y, z)}\mathbf{i} + \underbrace{N(x, y, z)}\mathbf{j} + \underbrace{P(x, y, z)}\mathbf{k}$

Given a curve C parametrized as $\mathbf{r}(t) = \underbrace{g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}}, \quad a \le t \le b$

1. Write $\underbrace{\mathbf{F}(g(t), h(t), k(t))} = M(g(t), h(t), k(t))\mathbf{i} + N(g(t), h(t), k(t))\mathbf{j} + P(g(t), h(t), k(t))\mathbf{k}$

   How? Rewrite the formulas for M, N, P shown here but replacing x, y, z with $g(t), h(t), k(t)$

   Call this $\mathbf{F}(\mathbf{r}(t))$

2. Compute $\frac{d\mathbf{r}}{dt}(t) = g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}$ from

3. Evaluate the 21B-style integral $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$

   The integrand of this integral is the dot product of two vectors:
   - $\mathbf{F}(\mathbf{r}(t))$ from Step 1
   - $\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t)$ from Step 2

Notation follows Thomas' Calculus: Early Transcendentals (12th Edition) as closely as possible