16.6 and 16.7: Surface Integrals, Orientability, and Stokes' Theorem[1]

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Flux across a surface

Let S be an orientable surface[2], whose orientation is given by $\mathbf{n}(x, y, z)$. Then the flux of $\mathbf{F}$ across the surface S is given by

$$\iint_S G(x, y, z)\, d\sigma$$

where $G(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z)$.

Finding n

- If S is described parametrically by $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, then
$$\mathbf{n} = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$
- If S is described implicitly with a function F of three inputs by $F(x, y, z) = 0$, then
$$\mathbf{n} = \pm \frac{\nabla F}{\|\nabla F\|}$$
- If S is described explicitly as points (x, y, z) such that $z = f(x, y)$ for a function f of two inputs, then
$$\mathbf{n} = \pm \frac{f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}}{\|f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}\|}$$

Stokes' Theorem

Let S be an orientable surface, whose orientation is given by $\mathbf{n}(x, y, z)$, and whose boundary is given by C, oriented counter-clockwise[3]. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

If $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$, then

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

[1] Notation follows Thomas' Calculus: Early Transcendentals (12th Edition) as closely as possible

[2] also called 'two-sided' or 'oriented'

[3] This is still ambiguous: what is meant is that C is oriented counter-clockwise from a "birds-eye view" where $\mathbf{n}$ points to "the bird/sky"