16.5 and 16.6: Surfaces, Area, and Surface Integrals

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Notation follows Thomas' Calculus: Early Transcendentals (12th Edition) as closely as possible

Parametric form

Suppose $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ with bounds $a \le u \le b$ and $c \le v \le d$ defines a surface S.

- **The region** R**:** the region in uv-space: $a \le u \le b$ and $c \le v \le d$.
- **Surface area differential:** $d\sigma = \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$.
- **Surface area integral:**

$$\iint_S d\sigma = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv = \int_c^d \int_a^b \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv.$$

- **General integral:** To integrate $G(x,y,z)$ over S,

$$\iint_S G(x,y,z)\, d\sigma = \iint_R G\Big(f(u,v), g(u,v), h(u,v)\Big)\, \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv.$$

Implicit form

S is the set of points (x,y,z) such that $F(x,y,z) = 0$ for some function $F(x,y,z)$

- **The region** R**:** the projection of the surface S onto the xy-plane (then $\mathbf{p} = \mathbf{k}$). Or, the projection of the surface S onto the xz-plane (then $\mathbf{p} = \mathbf{j}$). Or, the projection of the surface S onto the yz-plane (then $\mathbf{p} = \mathbf{i}$).
- **Surface area differential:** $d\sigma = \frac{\|\nabla F\|}{|\nabla F \cdot \mathbf{p}|}\, dA$. Here, $dA = dx\, dy$ if $\mathbf{p} = \mathbf{k}$.
- **Surface area integral:**

$$\iint_S d\sigma = \iint_R \frac{\|\nabla F\|}{|\nabla F \cdot \mathbf{p}|}\, dA.$$

- **General integral:** To integrate $G(x,y,z)$ over S,

$$\iint_S G(x,y,z)\, d\sigma = \iint_R G(x,y,z)\, \frac{\|\nabla F\|}{|\nabla F \cdot \mathbf{p}|}\, dA.$$

Explicit form

S is the set of points (x,y,z) such that $z = f(x,y)$. That is, we look at the graph of some function f.

- **The region** R**:** We'll usually need to pick a bounded subset of the domain space. That's R.
- **Surface area differential:** $d\sigma = \sqrt{{f_x}^2 + {f_y}^2 + 1}\, dx\, dy$
- **Surface area integral:**

$$\iint_S d\sigma = \iint_R \sqrt{{f_x}^2 + {f_y}^2 + 1}\, dx\, dy.$$

- **General integral:** To integrate $G(x,y,z)$ over S,

$$\iint_S G(x,y,z)\, d\sigma = \iint_R G(x,y,f(x,y))\, \sqrt{{f_x}^2 + {f_y}^2 + 1}\, dx\, dy.$$