The transportation problem and the diameters of transportation polytopes

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UW-L Mathematics Colloquium

Slides at `http://websites.uwlax.edu/ekim/talks/`

What is Linear Programming / Linear Optimization

Maximize the value of a given linear function $c : \mathbb{R}^n \to \mathbb{R}$ subject to a finite set of linear inequalities.

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\text{Maximize} & c(x, y, z) = 2x + 3y + 4z \\
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& 2x + 5y + 3z \leq 15 \\
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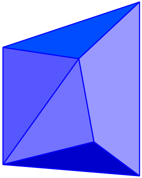
The Hirsch Conjecture

$\Delta(d,f)$ = maximum diameter among d-polytope with f facets



The Hirsch Conjecture

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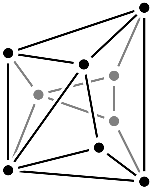

Conjecture [Hirsch 1957]

$$\Delta(d,f) \leq f - d$$

The Hirsch Conjecture

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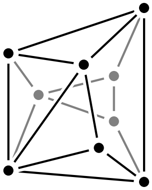


Ex-Conjecture [Hirsch 1957]

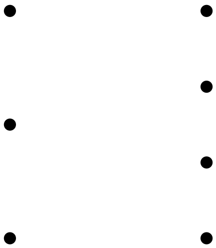
$$\Delta(d,f) \leq f - d$$

Theorem: Matschke, Weibel, Santos [2011]

$$\Delta(d,f) \geq \tfrac{21}{20}(f-d) \text{ for } d \geq 20$$

Transportation Problem

supplies **demands**



Transportation Problem

supplies			demands
Austin	•	•	Delhi
		•	Edmonton
Boston	•		
		•	Fukuoka
Charleston	•	•	Giza

Transportation Problem

supplies		**demands**	
Austin	8	9	Delhi
		1	Edmonton
Boston	8		
		3	Fukuoka
Charleston	6	9	Giza

Transportation Problem

supplies			demands
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Linear Program: the transportation problem

Given:

-

Transportation Problem

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Linear Program: the transportation problem

Given:

- the supply quantities

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Linear Program: the transportation problem

Given:

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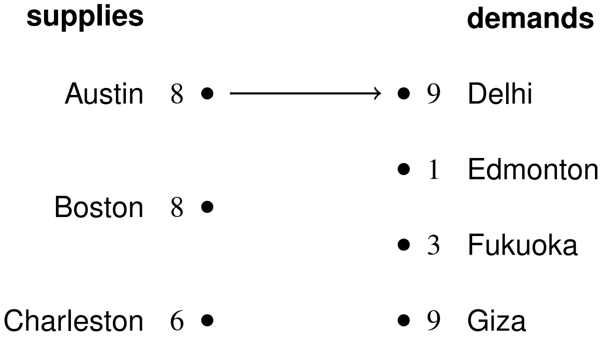
Transportation Problem

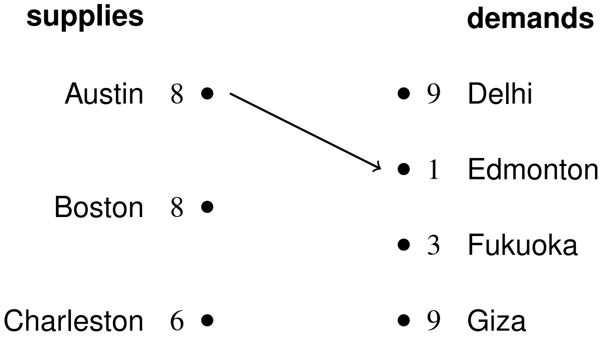


Linear Program: the transportation problem

Given:

- it costs \$9/unit to ship from Austin to Delhi

Transportation Problem

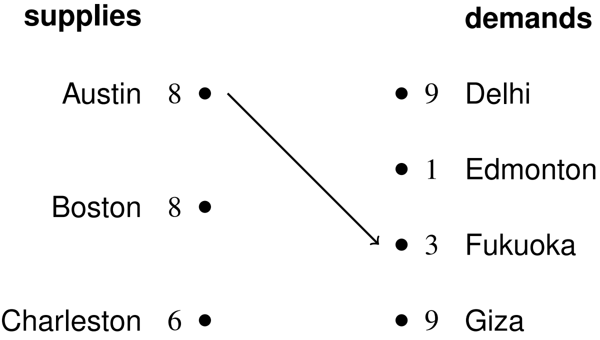


Linear Program: the transportation problem

Given:

- it costs $2/unit to ship from Austin to Edmonton

Transportation Problem

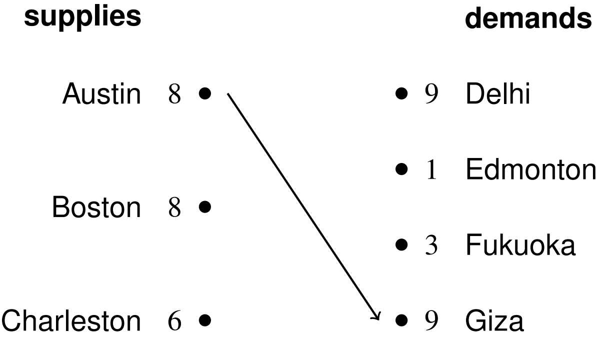


Linear Program: the transportation problem

Given:

- it costs \$7/unit to ship from Austin to Fukuoka

Transportation Problem

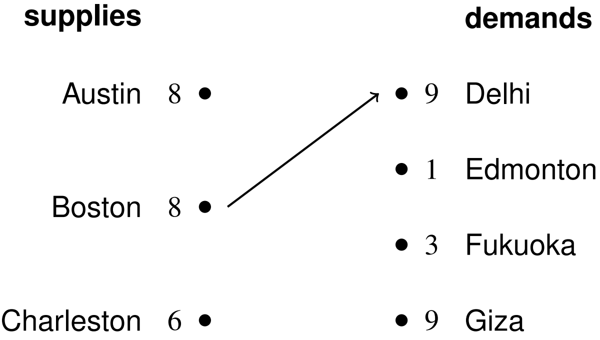


Linear Program: the transportation problem

Given:

- it costs $8/unit to ship from Austin to Giza

Transportation Problem

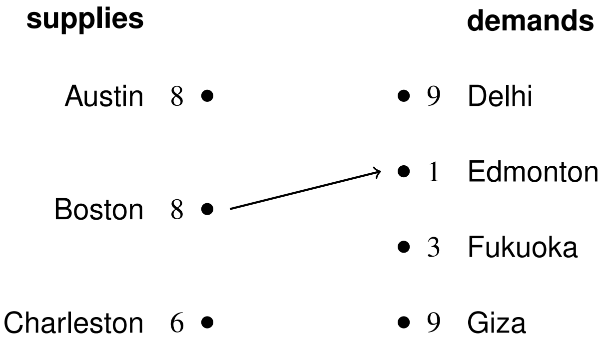


Linear Program: the transportation problem

Given:

- it costs \$8/unit to ship from Boston to Delhi

Transportation Problem

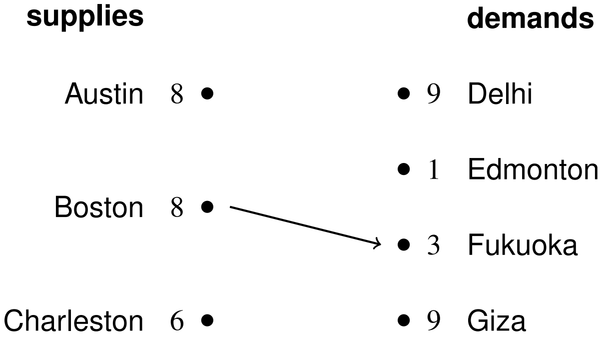


Linear Program: the transportation problem

Given:

- it costs $3/unit to ship from Boston to Edmonton

Transportation Problem



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Transportation Problem

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Austin	8	9	Delhi
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Linear Program: the transportation problem

Given:

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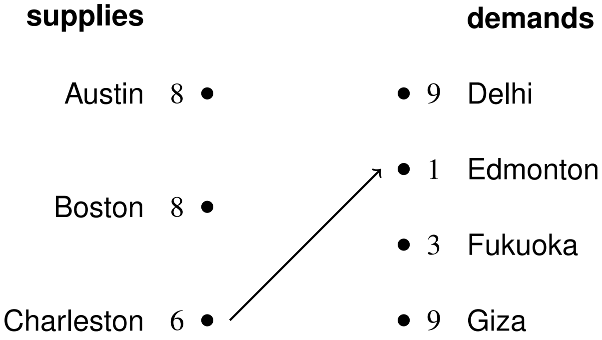
Transportation Problem

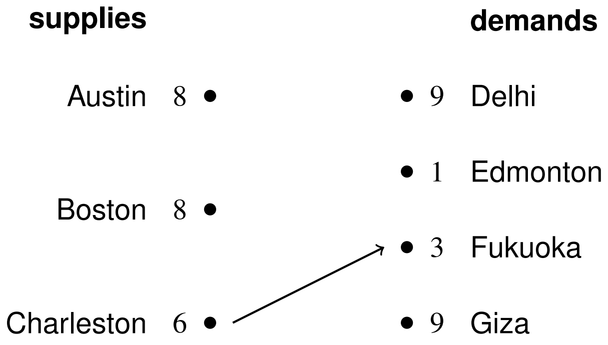


Linear Program: the transportation problem

Given:

- it costs \$4/unit to ship from Charleston to Edmonton

Transportation Problem



Linear Program: the transportation problem

Given:

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Transportation Problem

supplies		demands	
Austin	8	9	Delhi
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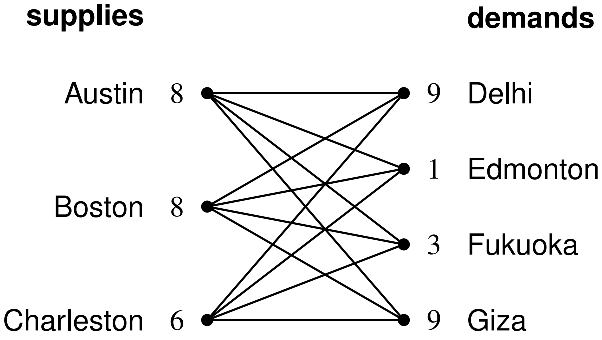
Linear Program: the transportation problem

Given:

- it costs \$6/unit to ship from Charleston to Giza

Transportation Problem



Linear Program: the transportation problem

Given:

- supply/demand amounts and per-unit transportation costs,

How much should be put on the plane from each supply to each demand to minimize the total cost of transporting?

Classical Transportation Polytopes

Let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ with positive entries.

Definition

The $m \times n$ **transportation polytope** determined by **margins** u and v is the set P of non-negative matrices $X = (x_{i,j})$ satisfying

$$\sum_{j=1}^{n} x_{i,j} = u_i \quad \forall i \qquad \text{and} \qquad \sum_{i=1}^{m} x_{i,j} = v_j \quad \forall j.$$

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	8
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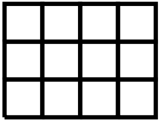

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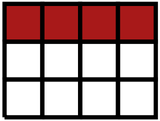

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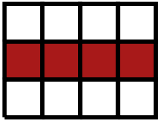

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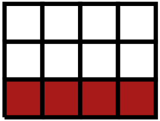

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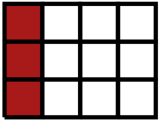

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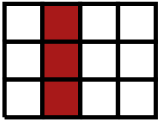

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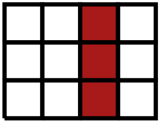

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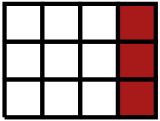

Hirsch-implied Bound

The $m \times n$ transportation polytope P defined by

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If the Hirsch Conjecture is true for transportation polytopes, then $\mathrm{diam}(P) \leq f - d \leq m + n - 1$.

Classical Transportation Polytopes

Example: $m = 3$, $n = 4$

If $u = (8, 8, 6) \in \mathbb{R}^m$ and $v = (9, 1, 3, 9) \in \mathbb{R}^n$,

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1	0	3	4	8
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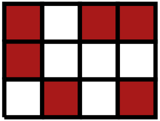

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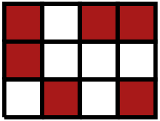

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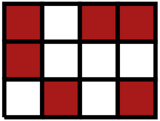

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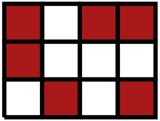
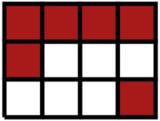
0	0	0	6	6
9	1	3	9	





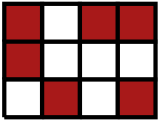
Classical Transportation Polytopes

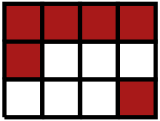
Example: $m = 3$, $n = 4$

If $u = (8, 8, 6) \in \mathbb{R}^m$ and $v = (9, 1, 3, 9) \in \mathbb{R}^n$, then the polytope P contains (among others), the following points:

1	0	3	4	8
8	0	0	0	8
0	1	0	5	6
9	1	3	9	

1	1	3	3	8
8	0	0	0	8
0	0	0	6	6
9	1	3	9	

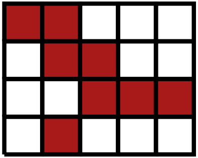




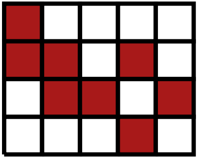
Both the point on the left and on the right happen to be vertices due to a Theorem of Klee and Witzgall.

Vertices with slice condition

Same slice contains the same single support entry



X'



Y'

Vertices X' and Y' have

Vertices with slice condition

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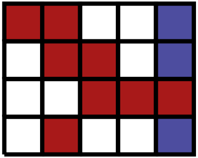
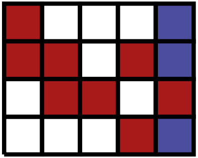


X'



Y'

Vertices X' and Y' have

- the same slice

Vertices with slice condition

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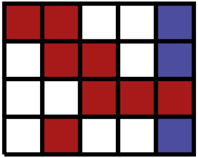

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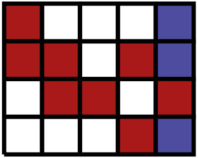


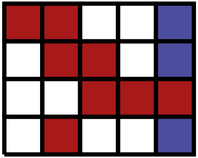
Y'

Vertices X' and Y' have

- the same slice
- which has a single support entry

Vertices with slice condition

Same slice contains the same single support entry



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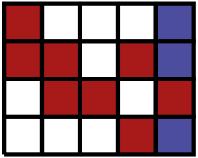

Y'

Vertices X' and Y' have

- the same slice
- which has a single support entry
- which is the ***same*** support entry

Quadratic Bound

Theorem: van den Heuvel, Stougie [2002]

Every $m \times n$ transportation polytope has diameter at most $(m+n)^2$.

Quadratic Bound

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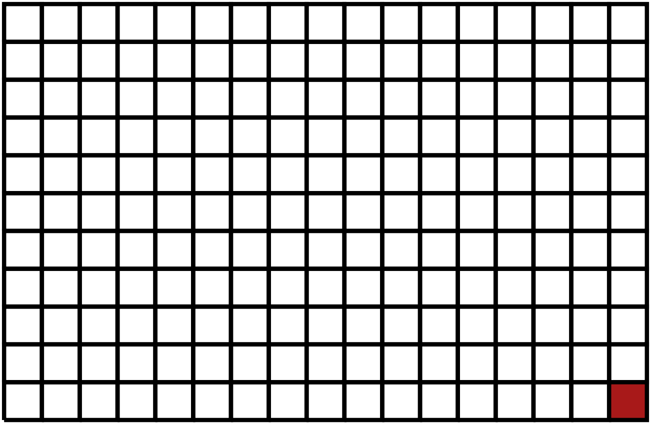
Lemma

Given two arbitrary vertices X and Y of an $m \times n$ transportation polytope P, there are vertices X' and Y' of P such that:

- The same slice in X' and Y' contains the same single support entry
- $\mathrm{dist}_P(X, X') + \mathrm{dist}_P(Y, Y') \leq 2(m+n-2)$.

Lemma: Proof Sketch



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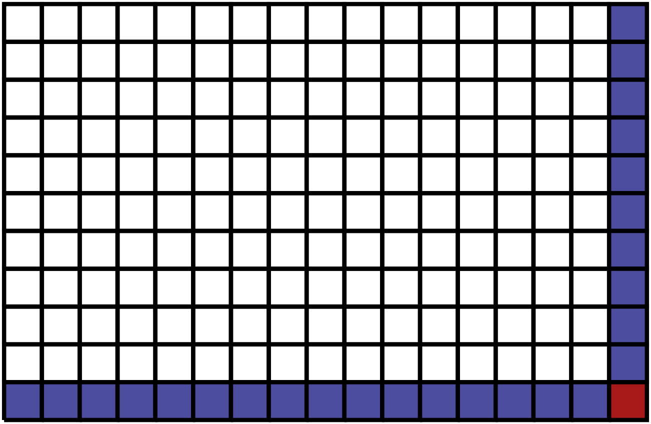

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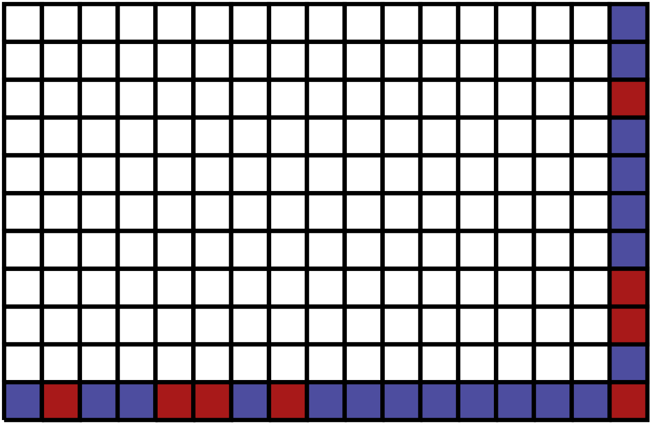

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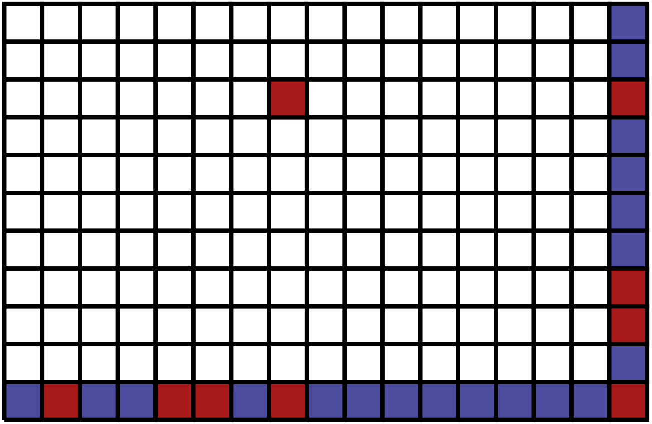

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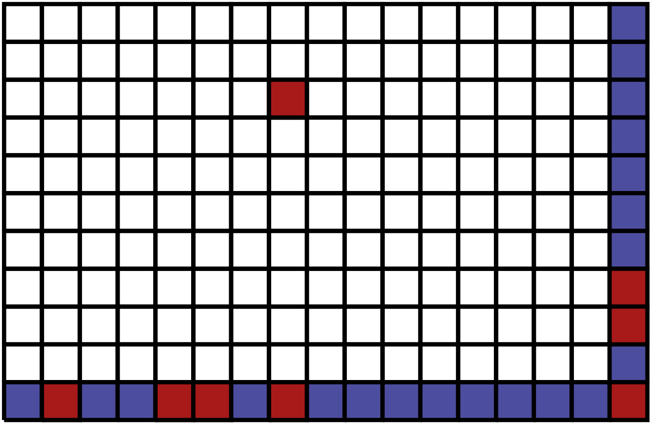

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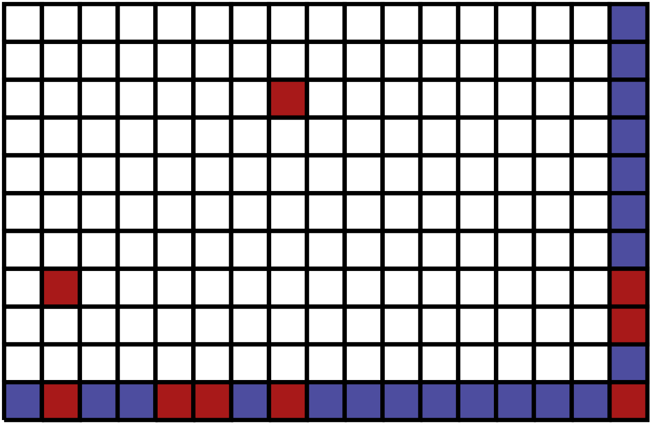

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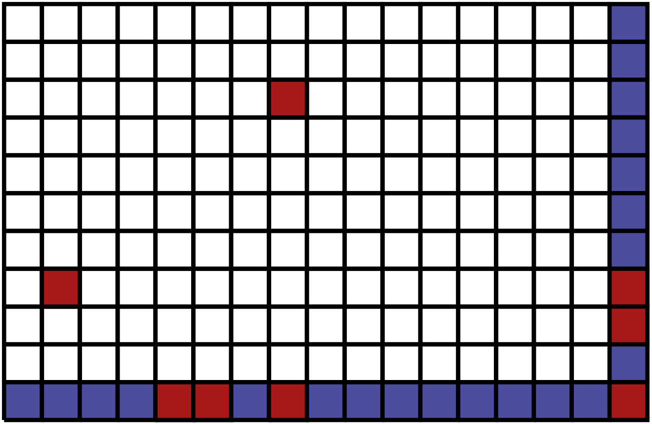

Linear Bound

Theorem: Brightwell, van den Heuvel, Stougie [2006]

Every $m \times n$ transportation polytope has diameter at most $8(m + n - 1)$.

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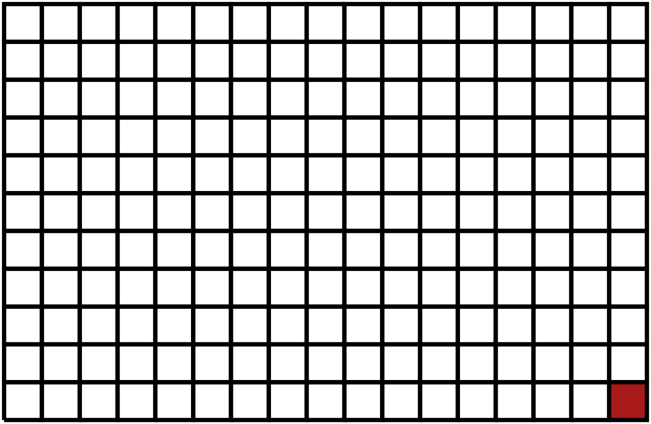
Every $m \times n$ transportation polytope has diameter at most $8(m+n-1)$.

Lemma

Given two arbitrary vertices X and Y of an $m \times n$ transportation polytope P, there are vertices X' and Y' of P such that:

- The same slice in X' and Y' contains the same single support entry
- $\mathrm{dist}_P(X, X') + \mathrm{dist}_P(Y, Y') \leq 8$.

Lemma: Proof Sketch



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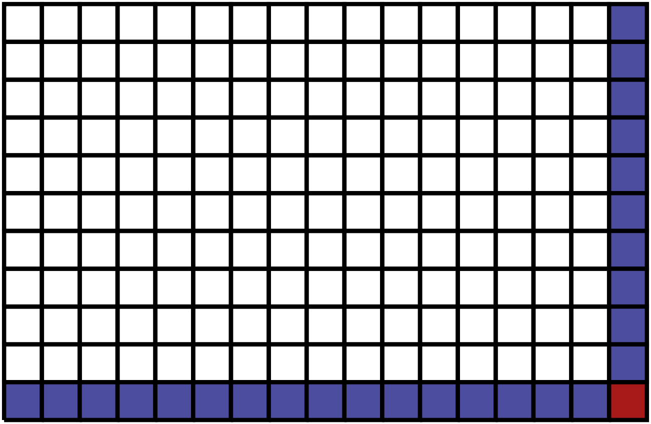


Better Linear Bound

Theorem: Hurkens [2009]

Every $m \times n$ transportation polytope has diameter at most $4(m + n - 1)$.

Better Linear Bound

Theorem: Hurkens [2009]

Every $m \times n$ transportation polytope has diameter at most $4(m+n-1)$.

Lemma

Given two arbitrary vertices X and Y of an $m \times n$ transportation polytope P, there is an integer $k > 0$ such that

- X' and Y' are vertices of P
- The same k slices in X' and Y' each contain the same single support entry
- $\mathrm{dist}_P(X, X') + \mathrm{dist}_P(Y, Y') \leq 4k$.

$2 \times n$ Transportation Polytopes

Theorem: De Loera, K. [2014]

Every $2 \times n$ transportation polytope satisfies the Hirsch Conjecture.

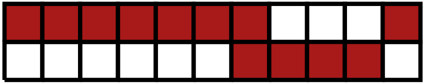
$2 \times n$ Transportation Polytopes

Theorem: De Loera, K. [2014]

Every $2 \times n$ transportation polytope satisfies the Hirsch Conjecture.

Lemma

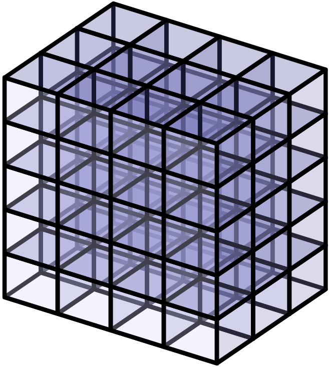
Every vertex X of a non-degenerate $2 \times n$ transportation polytope contains a unique column with two strictly positive coordinates.



3-way Transportation Polytopes

Definition

Given $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^p$,



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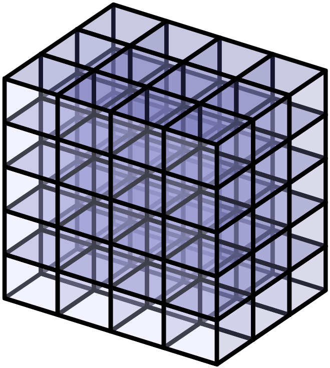

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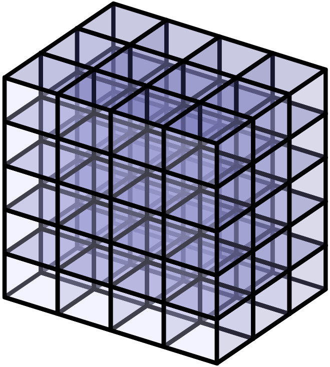

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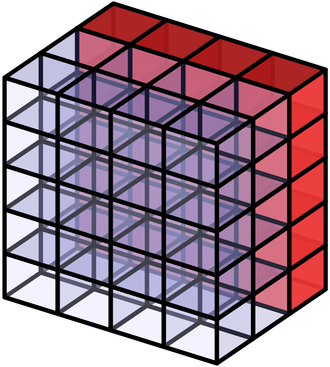

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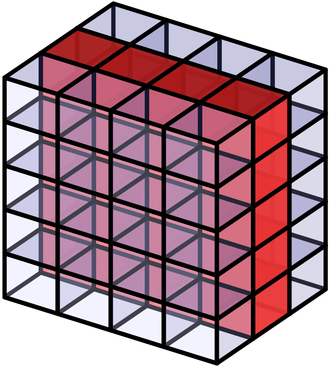

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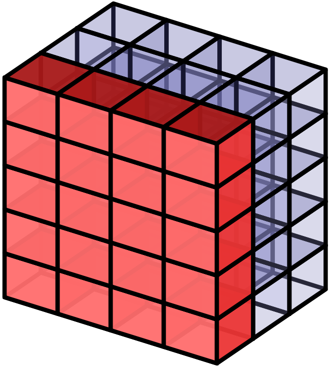

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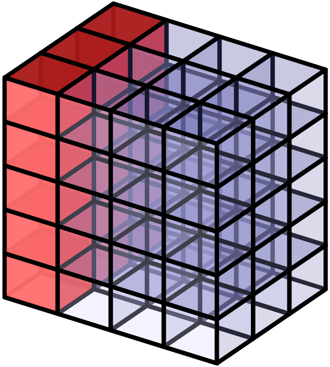

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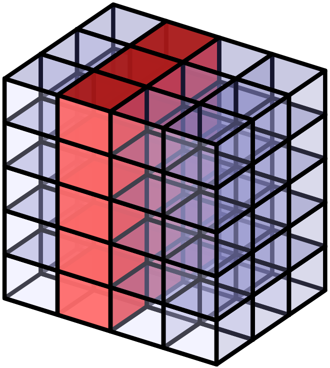

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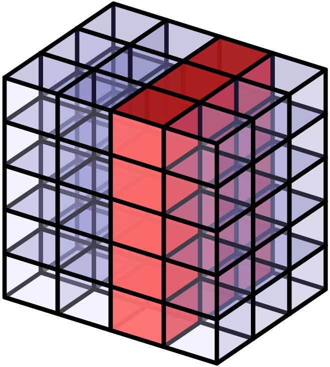

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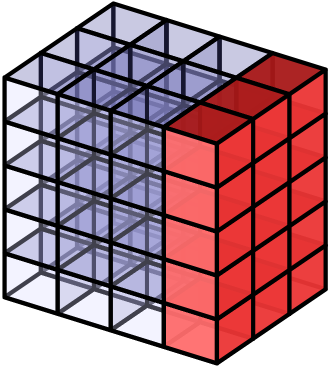

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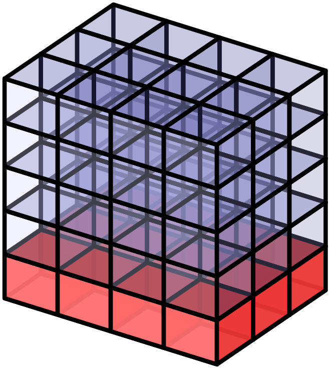

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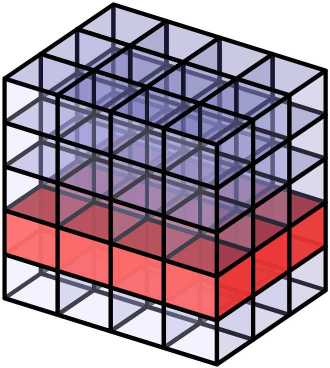

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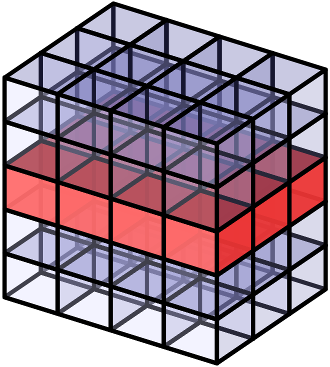

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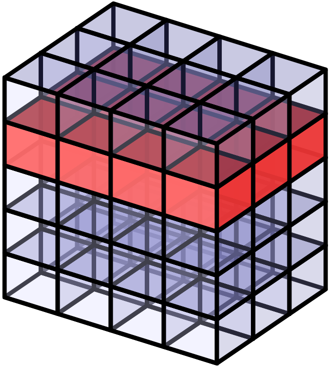

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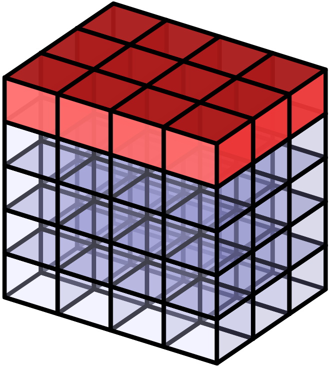
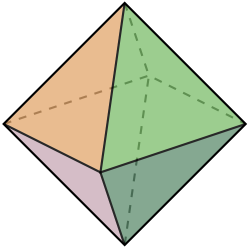
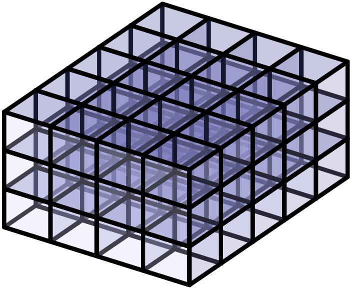


Significance of 3-way Transportation Polytopes



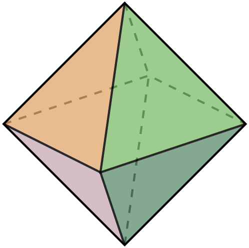
$\rightsquigarrow$



Theorem: De Loera, Onn [2006]

Given any rational polytope P,

Significance of 3-way Transportation Polytopes



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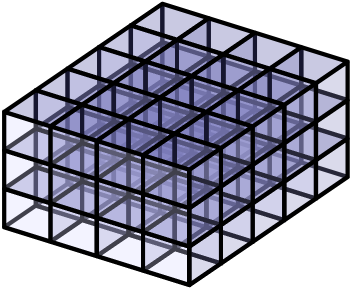

Theorem: De Loera, Onn [2006]

Given any rational polytope P,

- There is a 3-way transportation polytope Q given by 1-marginals with a face that is isomorphic to P.

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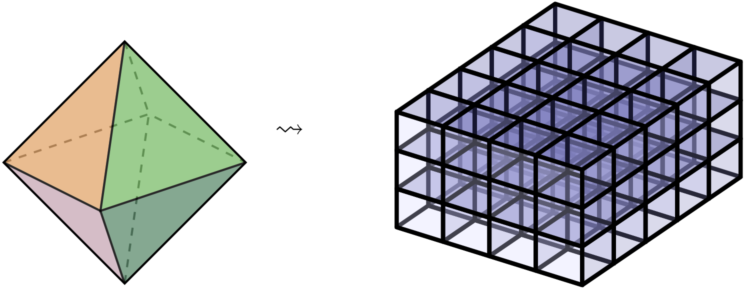
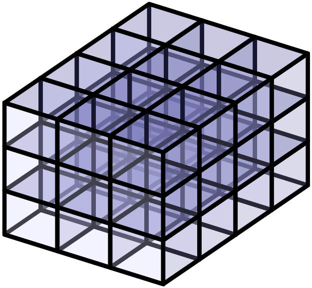
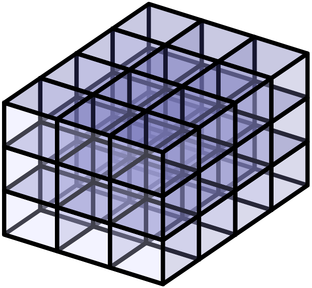


Theorem: De Loera, Onn [2006]

Given any rational polytope P,

- There is a 3-way transportation polytope Q given by 1-marginals with a face that is isomorphic to P.
- Moreover, the polytope Q can be computed in polynomial time (in the description of P).

“Same slice” with “same single support entry”

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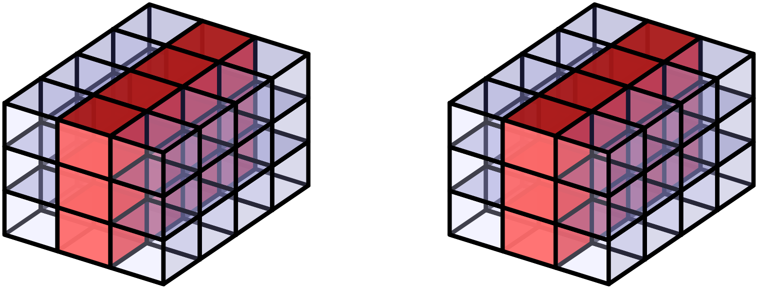

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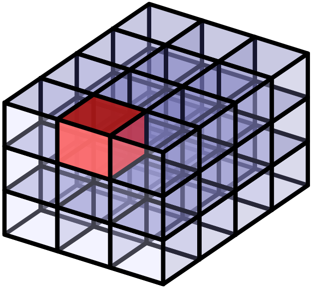

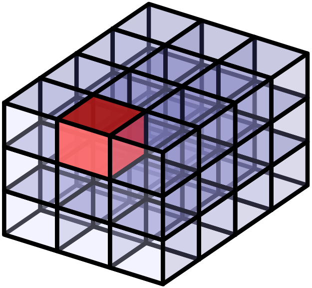

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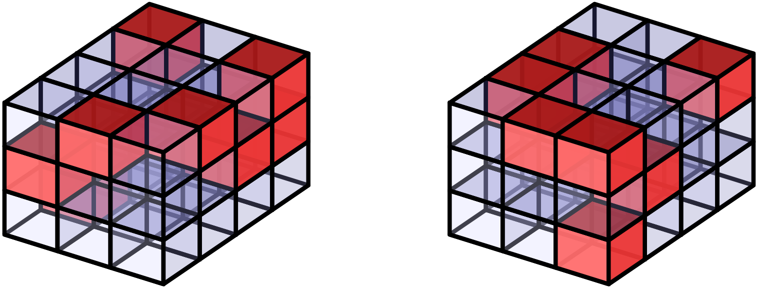

Quadratic Bound for Axial Transportation Polytopes

Theorem: De Loera, K., Onn, Santos [2009]

Every 3-way axial $m \times n \times p$ transportation polytope has diameter at most $2(m+n+p)^2$.

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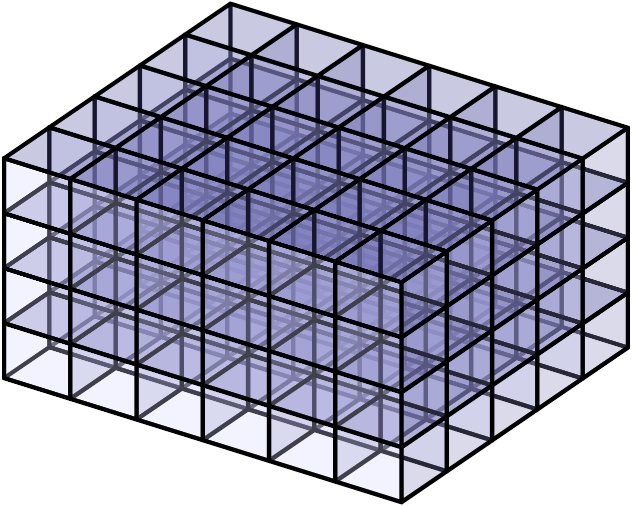
Lemma

Given two arbitrary vertices X and Y of an $m \times n \times p$ axial transportation polytope P, there are vertices X' and Y' of P such that:

- The same slice in X' and Y' contains the same single support entry
- $\mathrm{dist}_P(X, X') + \mathrm{dist}_P(Y, Y') \leq 4(m+n+p-1)$.

Lemma: Proof Sketch

Case 1



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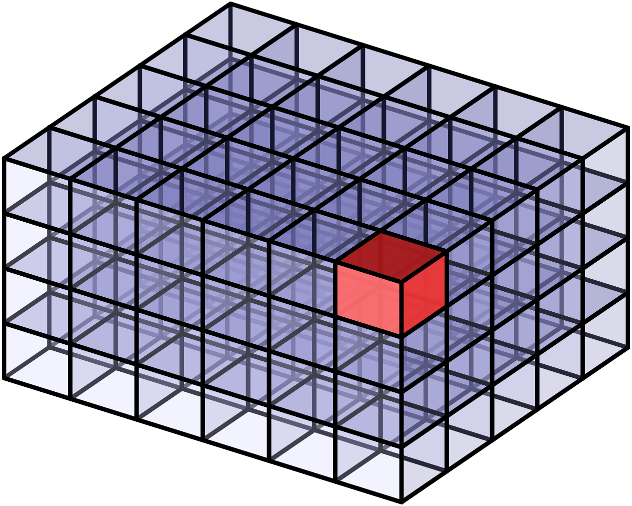

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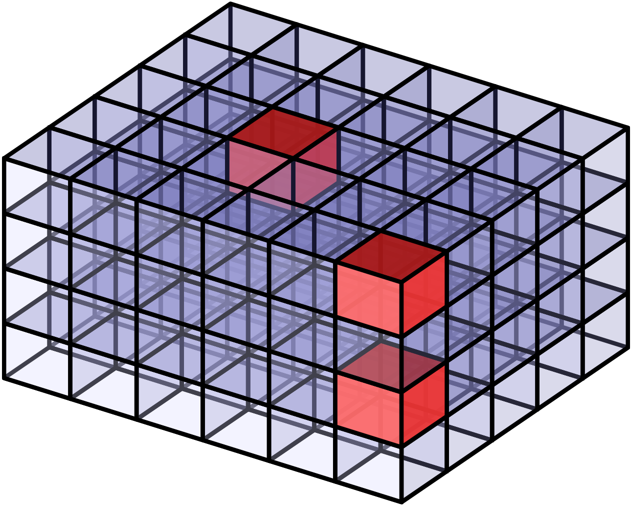

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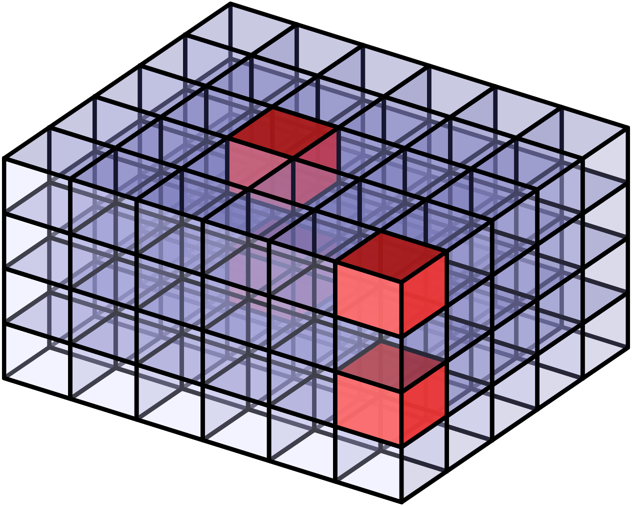
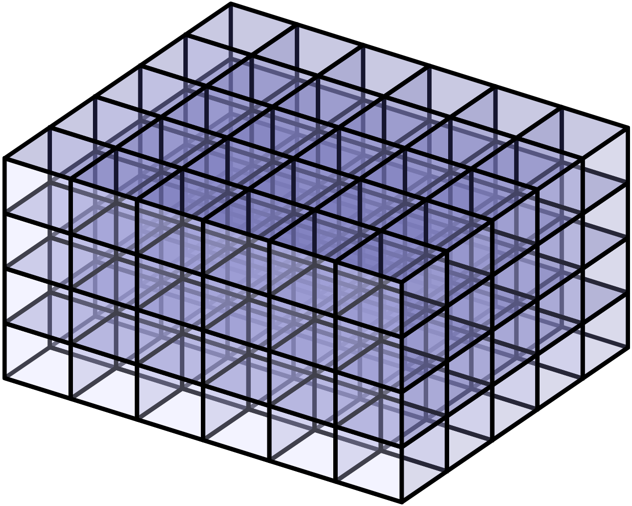


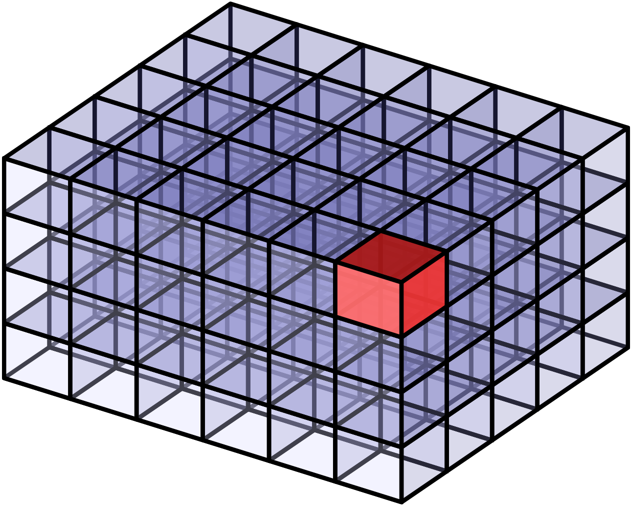
Lemma: Proof Sketch

Case 2



Lemma: Proof Sketch

Case 2



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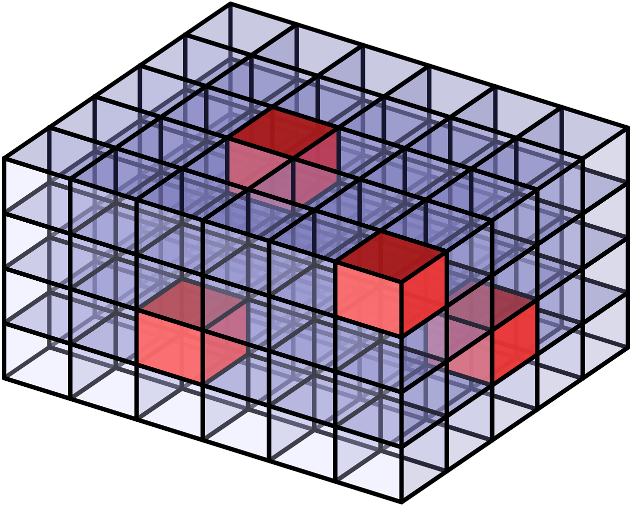

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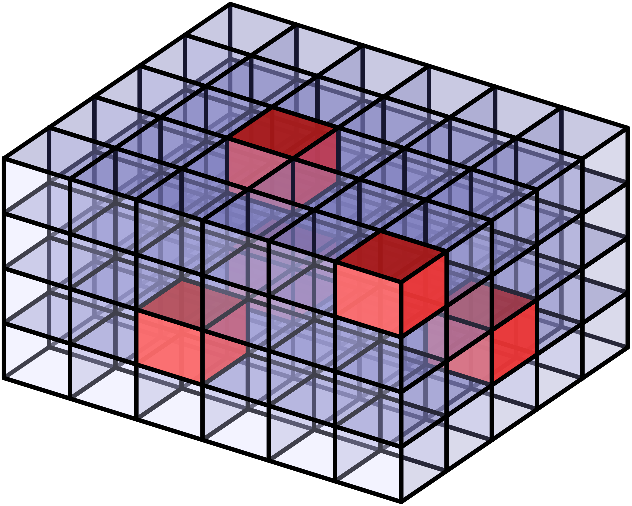

Thank you!

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