Exercise. Find the derivative of $g(x) = \int_3^x \sqrt{1+t^5}\,dt$ with respect to x.

- **Solution:** The integrand is $f(t) = \sqrt{1+t^5}$. By FTC1, $\frac{dg}{dx} = f(x) = \sqrt{1+x^5}$.

Exercise. Find the derivative of $h = \int_7^x e^{-t^2}\,dt$ with respect to x.

- **Solution:** The integrand is $f(t) = e^{-t^2}$. By FTC1, $\frac{dg}{dx} = e^{-x^2}$.

Fundamental Theorem of Calculus 1, rewritten with u Let's just assume that $f(t)$ is continuous everywhere (just to have fewer technicalities).

Fix a number called a. Define a function g with input u by $g = \int_a^u f(t)\,dt$. Then, the derivative of g with respect to u is given by

$$\frac{dg}{du} = f(u).$$

Exercise. Find the derivative of $k = \int_7^u e^{-t^2}\,dt$ with respect to u using the rewritten FTC1.

- The integrand is $f(t) = e^{-t^2}$. By FTC1, $\frac{dk}{du} = e^{-u^2}$.

Chain Rule Formula, rewritten

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}.$$

Exercise. Find the derivative of $g(x) = \int_1^{x^4} \sec t\,dt$ with respect to x.

- **Solution:** Let $u = x^4$. Then, g can be rewritten as $g = \int_1^u \sec t\,dt$. We were asked to find $\frac{dg}{dx}$. By the Chain Rule,

  $$\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx},$$

  so we need to find $\frac{dg}{du}$ and $\frac{du}{dx}$.

  Finding $\frac{dg}{du}$ is just like the set up of the previous exercise. By using the FTC1 rewritten with the u,

  $$\frac{dg}{du} = \sec u.$$

  and $\frac{du}{dx}$ is more routine:

  $$\frac{du}{dx} = 4x^3.$$

  Putting this all together:

  $$\frac{dg}{dx} \overset{\text{Chain}}{=} \frac{dg}{du} \cdot \frac{du}{dx} = \underbrace{\Big(\sec u\Big)}_{\frac{dg}{du}} \underbrace{\Big(4x^3\Big)}_{\frac{du}{dx}} = \sec(x^4) \cdot 4x^3.$$

Exercise. Find the derivative of $g(x) = \int_9^{\sin x} e^{-t^2}\,dt$ with respect to x.

- **Solution:** Let $u = \sin x$. Then g can be rewritten as $g = \int_9^u e^{-t^2}\,dt$. Like the last problem,

$$\frac{dg}{dx} = \underbrace{\left(e^{-u^2}\right)}_{\frac{dg}{du}}\underbrace{\left(\cos x\right)}_{\frac{du}{dx}} = e^{-(\sin x)^2}\cdot\cos x.$$

which can also optionally be rewritten as

$$e^{-\sin^2 x}\cos x \quad \text{or} \quad \frac{\cos x}{e^{\sin^2 x}}.$$

Exercise. Find the derivative of $g(x) = \int_{\log_3 x}^{20} \sqrt{1+\cos t}\,dt$ with respect to x.

- **Solution:** The function is the lower endpoint of integration, so rewrite g as follows:

$$g = -\int_{20}^{\log_3 x} \sqrt{1+\cos t}\,dt.$$

Things can start to get confusing with the minus sign, so I'd like to set up a new variable to represent all of what's in g except the minus sign and I'll call it h. To be a bit clearer:

$$g = -\underbrace{\int_{20}^{\log_3 x} \sqrt{1+\cos t}\,dt}_{h}.$$

In other words,

$$h = \int_{20}^{\log_3 x} \sqrt{1+\cos t}\,dt.$$

so that $g = -h$. And so $\frac{dg}{dx} = -\frac{dh}{dx}$.
Using $u = \log_3 x$, rewrite h like this:

$$h = \int_{20}^{u} \sqrt{1+\cos t}\,dt.$$

Thus,

$$\frac{dh}{dx} \overset{\text{Chain}}{=} \frac{dh}{du}\cdot\frac{du}{dx} = \underbrace{\left(\sqrt{1+\cos u}\right)}_{\frac{dh}{du}}\underbrace{\left(\frac{1}{x\ln 3}\right)}_{\frac{du}{dx}} = \frac{\sqrt{1+\cos(\log_3 x)}}{x\ln 3}.$$

To finish, recall we already said that $\frac{dg}{dx} = -\frac{dh}{dx}$. Thus,

$$\frac{dg}{dx} = -\frac{dh}{dx} = -\frac{\sqrt{1+\cos(\log_3 x)}}{x\ln 3}.$$

Exercise. Find the derivative of $g(x) = \displaystyle\int_{\log_3 x}^{x^6} \sqrt{1+\cos t}\,dt$ with respect to x.

- **Solution:** The integral has *both* endpoints of integration being functions, so we need to rewrite g by splitting the integral into two separate integrals:

$$\begin{aligned} g &= \int_{\log_3 x}^{x^6} \sqrt{1+\cos t}\,dt \\ &= \int_{\log_3 x}^{12345} \sqrt{1+\cos t}\,dt \quad + \quad \int_{12345}^{x^6} \sqrt{1+\cos t}\,dt \\ &= \int_{12345}^{x^6} \sqrt{1+\cos t}\,dt \quad + \quad \int_{\log_3 x}^{12345} \sqrt{1+\cos t}\,dt \\ &= \underbrace{\int_{12345}^{x^6} \sqrt{1+\cos t}\,dt}_{h} \quad - \quad \underbrace{\int_{12345}^{\log_3 x} \sqrt{1+\cos t}\,dt}_{k}. \end{aligned}$$

Since $g = h - k$, the using the Difference Rule for differentiation, $\dfrac{dg}{dx} = \dfrac{dh}{dx} - \dfrac{dk}{dx}$.

Let $u = x^6$ and $v = \log_3 x$. (Note that I'm using two **DIFFERENT** letters: u and v, just for good bookkeeping.) Using these choices for u and v, we can rewrite both h and k. They are rewritten like this:

$$h = \int_{12345}^{u} \sqrt{1+\cos t}\,dt \qquad \text{and} \qquad k = \int_{12345}^{v} \sqrt{1+\cos t}\,dt.$$

Then,

$$\frac{dg}{dx} = \frac{dh}{dx} - \frac{dk}{dx} = \underbrace{\frac{dh}{du}\frac{du}{dx}}_{\frac{dh}{dx}} - \underbrace{\frac{dk}{dv}\frac{dv}{dx}}_{\frac{dk}{dx}} = \underbrace{\left(\sqrt{1+\cos u}\right)}_{\frac{dh}{du}}\underbrace{\left(6x^5\right)}_{\frac{du}{dx}} - \underbrace{\left(\sqrt{1+\cos v}\right)}_{\frac{dk}{dv}}\underbrace{\left(\tfrac{1}{x\ln 3}\right)}_{\frac{dv}{dx}}$$

All that's left to do is replace u and v, since we introduced them:

$$\frac{dg}{dx} = \left(\sqrt{1+\cos(x^6)}\right)\left(6x^5\right) - \left(\sqrt{1+\cos(\log_3 x)}\right)\left(\tfrac{1}{x\ln 3}\right)$$

Exercise. Find the derivative of $g(x) = \int_x^{5^x} \sqrt{3+\ln t}\,dt$ with respect to x.

- **Solution:** First rewrite g. Since there's an x in both endpoints of integration, we'll again have to rewrite the integral as the sum of two integrals.

$$\begin{aligned} g &= \int_x^{5^x} \sqrt{3+\ln t}\,dt \\ &= \int_x^{13.17} \sqrt{3+\ln t}\,dt \quad + \quad \int_{13.17}^{5^x} \sqrt{3+\ln t}\,dt \\ &= \underbrace{\int_{13.17}^{5^x} \sqrt{3+\ln t}\,dt}_{h} \quad - \quad \underbrace{\int_{13.17}^{x} \sqrt{3+\ln t}\,dt}_{m} \end{aligned}$$

Rewrite h by using $u = 5^x$. Then we get:

$$h = \int_{13.17}^{u} \sqrt{3+\ln t}\,dt$$

Note that because m is an integral which already has an x at the top, **there is no need** to replace the x with a v.

Since $g = h - m$, we have

$$\frac{dg}{dx} = \frac{dh}{dx} - \frac{dm}{dx} = \underbrace{\frac{dh}{du}\cdot\frac{du}{dx}}_{\frac{dh}{dx}} - \frac{dm}{dx} = \underbrace{\left(\sqrt{3+\ln u}\right)}_{\frac{dh}{du}}\underbrace{\left(5^x \ln 5\right)}_{\frac{du}{dx}} - \underbrace{\sqrt{3+\ln x}}_{\frac{dm}{dx}} = \left(\sqrt{3+\ln(5^x)}\right)\left(5^x\ln 5\right) - \sqrt{3+\ln x}$$

How many variable substitutions are required? Note again that there is NO NEED to rewrite m by saying that $v = x$ and then writing

$$m = \int_{13.17}^{v} \sqrt{3+\ln t}\,dt$$

It is by no means wrong to do this, but note that you'll write $\frac{dm}{dx} = \frac{dm}{dv}\cdot\frac{dv}{dx}$ and that $\frac{dv}{dx}$ is just 1 anyway! Again, it's not WRONG, but this is a waste of time!

More importantly, if you just do this substitution without thinking, then you've sort of missed the point of *why* we did these substitutions in the first place!! Don't turn this into a routine. The use of u had a purpose! For example, go back to the first three exercises:

- Find the derivative of $g(x) = \int_3^x \sqrt{1+t^5}\,dt$ with respect to x.
- Find the derivative of $h = \int_7^x e^{-t^2}\,dt$ with respect to x.
- Find the derivative of $k = \int_7^u e^{-t^2}\,dt$ with respect to u.

You did not do a variable substitution in ANY of these! (You did not need the Chain Rule for any of these!) In same way, look at m. If the FIRST exercise on this practice sheet was

- Find the derivative of $m = \int_{13.17}^x \sqrt{3+\ln t}\,dt$ with respect to x

you would NOT have said $u = x$ or $v = x$.

Exercise. Find the derivative of $h(z) = \tan\left(\int_8^{z^6} \sqrt{1+m^2}\, dm\right)$ with respect to z.

- **Solution:** Let $g = \int_8^{z^6} \sqrt{1+m^2}\, dm$ so then $h = \tan(g)$. The question asks us to find $\frac{dh}{dz}$. By the Chain Rule,
$$\frac{dh}{dz} = \frac{dh}{dg}\frac{dg}{dz}.$$

Since we need to find $\frac{dg}{dz}$, let's do that first. Note that where there is traditionally a t, there is now an m. Where there's traditionally an x, there's currently a z. But all the same ideas still apply, just with different letters! Let $u = z^6$. We can rewrite g like this:

$$g = \int_8^{u} \sqrt{1+m^2}\, dm$$

So

$$\frac{dg}{dz} = \frac{dg}{du}\frac{du}{dz} = \underbrace{\left(\sqrt{1+u^2}\right)}_{\frac{dg}{du}}\underbrace{\left(6z^5\right)}_{\frac{du}{dz}} = \left(\sqrt{1+(z^6)^2}\right)\left(6z^5\right)$$

And thus,

$$\frac{dh}{dz} = \frac{dh}{dg}\frac{dg}{dz} = \underbrace{\left(\sec^2(g)\right)}_{\frac{dh}{dg}}\underbrace{\left((\sqrt{1+(z^6)^2})(6z^5)\right)}_{\frac{dg}{dz}} = \left[\sec^2\left(\int_8^{z^6} \sqrt{1+m^2}\, dm\right)\right] \cdot \left[(\sqrt{1+(z^6)^2})(6z^5)\right]$$

Ensure that your final answer looks like this. We introduced the g and u, so the final answer shouldn't have any of these. The final answer should only be in terms of z and m.