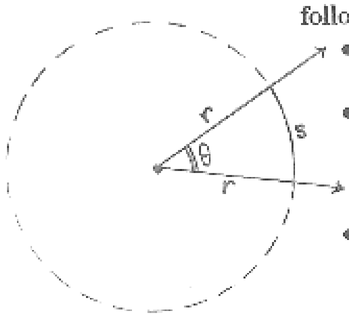
PHY 209
Space and Time in Elementary Physics

Euclidean Geometry: Angles, Lengths, Areas, Volumes

Angles

What is an angle? An angle is formed when two rays radiate from a common point (called the <u>vertex</u> of the angle). (This occurs, for instance, when two lines intersect at a point.) The **angle θ between the two rays** is defined by following procedure.



- Draw a circle of *any* radius r, centered at the vertex.
- Now, the two rays cut the circle into two arcs. Measure the length of the arc corresponding to the angle you are interested in. Call this length s.
- The <u>angle</u> is:

$$\text{The } \textbf{angle } \theta = \frac{s}{r} = \frac{\text{arc-length}}{\text{radius}} \text{ (in radians).}$$

A <u>radian</u> is a unit of angular measure, just like the degree. The radian, however, is a more natural unit to use. Let's see why.

- Measure the angle (in radians) of a complete circle by measuring the arc-length of a complete circle and then dividing this by the radius. (Hint: think about the first assignment.)

What is another name for "the arc-length of a complete circle"?

Ans:

- What is the angle (in radians) of a complete circle?

Ans:

You should have gotten a really nice formula in terms of π, which we learned was a special constant that was naturally associated with Euclidean geometry. (That's what the first project taught us!)

Now, we know that the angle (in degrees) of a complete circle is 360 degrees. But where did the number 360 come from? Nowhere special. It just seemed to be convenient because it is a multiple of 2, 3, 4, 5, 6, (and, of course, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180). By this sort of method, we could have chosen some other number if we wished. There's nothing special about 360... but, as we've already learned, there is something very special about π.

- What is 1 radian measured in degrees?

Ans:

- With only a piece of string and a circle of any radius, (*i.e.*, without a protractor,) construct an angle of 1 radian. Clearly explain what you did.
  Ans:

Lengths

Unlike an angle, which has a natural unit (namely, the radian), there is no natural unit for length. So, what one must do is to define a **standard** unit of length.

- Take a piece of string of some length. This is could be a standard of length. So, let's call the unit defined by this string "1 segment".

- Only using this standard, construct a line whose length is 3 segments. Explain how you did this.
  Ans:

- Construct a line whose length is $\frac{1}{3}$ segments.
  Explain how you did this.
  Ans:

- Construct a line whose length is $\frac{3}{4}$ segments.
  Explain how you did this.
  Ans:

As you know, the **inch** is a standard unit of length in the **English system** of units. So is the **foot** and the **yard**. Other English units of length, like the **furlong** and the **mile**, are probably defined in terms of one of the others. Do you remember all of the *(awful)* unit-conversions between them?

- Write down as many conversions among the English units as you can.
  Ans:

In the **Metric System** (*i.e.*, in **SI units**), the **meter** is the standard unit of length. As you know, all of units of length in this system have very simple unit-conversions among them. You should become familiar with the following commonly used metric prefixes in physics and elsewhere: **nano**$=10^{-9}$, **micro**$=10^{-6}$, **milli**$=10^{-3}$, **centi**$=10^{-2}$, **kilo**$=10^{3}$. Because this system is so simply organized (as opposed to the English system), it is a preferred system of units in physics and other sciences.

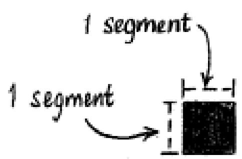
You might be wondering how the meter was first defined. Well, one early definition of the meter was that it was $\frac{1}{10{,}000{,}000}$ the distance along the Earth's surface from the North Pole to the Equator.

- Using this standard, calculate the polar circumference and the polar radius of the Earth (as opposed to their equatorial counterparts). You might find it helpful to draw a picture.

Ans:

Over the years, the meter was redefined to be the length of particular metal bar in France, then later the wavelength of radiation from a particular isotope of Krypton, and now finally a certain fraction of the distance traveled by light in 1 sec.

Later, we will develop methods to measure lengths indirectly (*i.e.*, without putting a ruler right up against the length we are trying measure). But we need some more mathematical tools first.

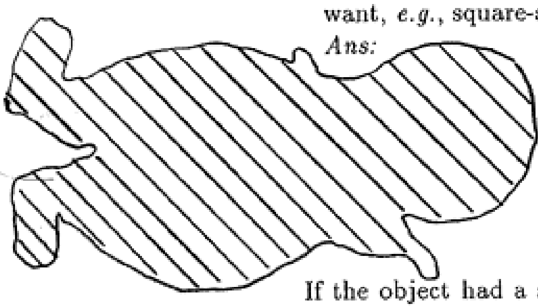


Areas

Just like for lengths, one needs a standard unit of area. A convenient standard would be a square, each of whose edges has the length of 1 segment (the length of your standard unit of length). Thus, the area of your standard would be 1 square-segment.

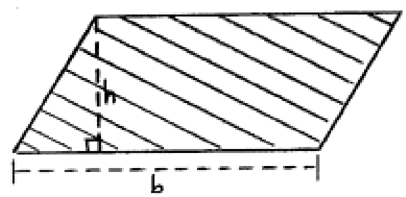
- If you were given an irregular object (like the one below), how would you measure its area? Invent as many methods as you can. Choose one method to determine the area of this object (in any units you want, *e.g.*, square-segments, in^2, cm^2, km^2).

  Ans:



If the object had a simpler shape, you could apply formulas that could be expressed in terms, of say, the length of certain edges or line segments. For instance, if I had a **rectangle** with base of length b and height of length h, then the area of this rectangle is, of course, $A = bh$. It turns out that this is also the area of a **parallelogram** with base b and height h, where the height h of a parallelogram is (perpendicular) length between the base and the parallel side opposite it. Observe that a rectangle is a special case of a parallelogram.

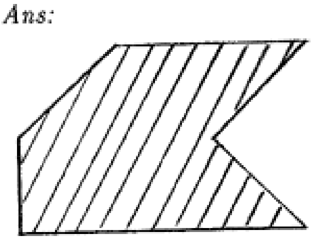
- Using a geometrical construction, show that the **area of a parallelogram** is $A = bh$.
  Ans:



- Using this result, explain why the **area of a triangle** is $A = \frac{1}{2}bh$.
  Ans:

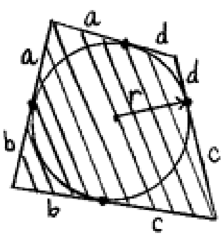
- Suppose I doubled the length of all of the sides of the above parallelogram. What is the area of the new parallelogram (as compared with the original parallelogram)?
  Ans:

- Find the area of the following object (in whatever units you want). Explain your method.
  Ans:



A neat formula that I discovered on my own one day, which is probably well-known to mathematicians, is that if you can *inscribe* a circle within a polygon

$P = (a+b)+(b+c)+(c+d)+(d+a)$



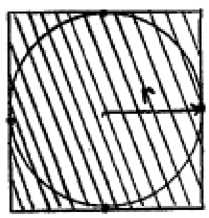
(in other words, if you can draw a circle inside a polygon which touches each side of the polygon at one point) then the area of the polygon is

$$A = \frac{1}{2}(\text{radius of that circle})(\text{perimeter of the polygon}) = \frac{1}{2} rP.$$

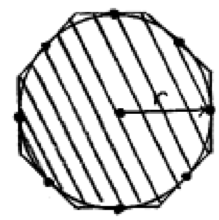
It turns out that the polygon *need not be regular* (*i.e.*, it need not have all sides equal and all angles equal).

- Check this formula for the square below.

  Ans:

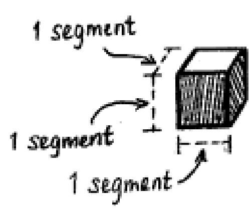


Observe that if I use a regular octagon and higher-n gons that these polygons approximate that inscribed circle better and better.



- Taking the "perimeter of a circle" to be its circumference, derive the formula for the area of a circle.

  Ans:



Volumes

By now, you should have guessed that a cube, with each edge 1 segment long, would be a convenient standard of volume. Thus, the volume of your standard would be 1 cubic-segment.

- If you were given an irregular solid (say, a hammer), how would you measure its volume? Invent as many methods as you can.

  Ans:

(Optional reading) You might think it is easy to measure the length of something. There is a famous question *"What is the length of the coast of Britain?"* Well, on a map, it seems like one can lay out a piece of string around the coast and determine a length (after appropriately scaling). After all, that's the best you could have done. But suppose you were actually sent there to measure it. You would find that the coast is not as smooth as the map suggested. Instead, it's quite rough and jagged. If you now take your standard string above and measure the coast as best as you can, you would get a larger number (after appropriately converting between the system of units). It seems, in fact, the smaller your measuring instrument, the longer the length you'll measure. Notice, though that if you measure the area of Britain with the associated area-measuring instrument that the area isn't getting that much bigger. In other words, the area is really finite but (I think) the length of coastal boundary is infinite. You might have guessed that this sort of question comes up in the discussion of **fractals**. We don't have the time or the need (and I don't know that much more about it) to discuss this further in this class. I just thought you might find this interesting. If you are interested in learning more about this, speak to me after class... I'll try to find something for you.

Bonus Question (1 pt):

Take an ordinary sheet of paper. Can you cut a hole in it in such a way that you can completely step through it?