PHY 209
Space and Time in Elementary Physics

Solving Equations

Galileo's "Law of Falling Bodies"

Galileo Galilei was an Italian scientist in the 1600s. Among his many contributions to science (*e.g.*, the invention of the telescope and the subsequent discovery of the large moons of Jupiter), he formulated the celebrated **"Law of Falling Bodies"**:

All bodies fall with the same constant acceleration.

(It is usually understood that this always refers to ideal situations in which there is, *e.g.*, no air resistance.) *(This means that: neglecting air resistance, a feather and a stone, dropped from the same height, will reach the ground at the same time.)*

One experiment that led Galileo to formulate such a law is the timing balls as they rolled down inclined-planes (*i.e.*, "ramps"). His result was that when the balls are released from rest at the top of the incline,
the displacement traveled was proportional to the square of the elapsed-time.
In other words, that

$$x = kt^2, \text{ where } k \text{ is some constant.}$$

It will turn out that this constant k is proportional to the acceleration.

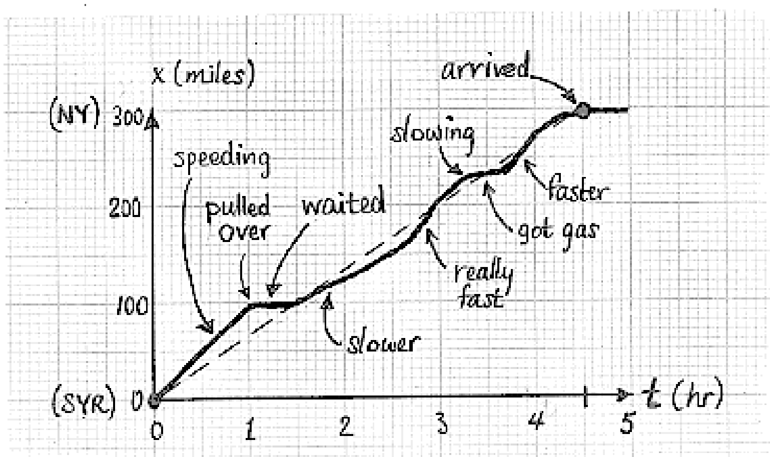
In class, we will try to carry out our version of Galileo's experiment. We will roll a wheel down an inclined-plane. We have a motion-detector set-up to measure the wheel's position x, as a function of time t. A computer will collect the data points and display them on a screen. We will do a crude analysis to check Galileo's result.

Some Definitions

Picture a ruler, and imagine its extension to accomodate "negative values". Note the location of the origin (*i.e.*, the "zero mark"). The **position** of an object located along the ruler is **"its displacement from the origin"**. Displacement takes "direction" into account. For one-dimensional motion, this means that displacement could be positive, negative, or zero. By contrast, "distance" is the magnitude (*i.e.*, absolute-value) of the displacement—distance neglects direction.

The **velocity** of an object is its **"rate of change of position with time"**. The velocity is also a directional-quantity. For one-dimensional motion, the velocity could be positive, negative, or zero. By contrast, "speed" is the magnitude (*i.e.*, absolute-value) of the velocity—speed neglects direction. For a car, the speed is measured by (you guessed it!) its speedometer. *[On a graph of position-vs.-time, velocity is* **"the slope of the curve"**. *We will study this interpretation more carefully when we study calculus.]* Velocity is a measure of "how fast you are going **at that instant**, and in what direction".

By contrast, the **AVERAGE-velocity** of an object **OVER AN INTERVAL** is a "smoothed" measure of "how fast you were traveling during some period of time". Suppose you started traveling from Syracuse at noon, and arrived in New York, 300 miles away, at 4:30pm. Your AVERAGE-velocity would be $\frac{300}{4.5} = 66.6$ mi/hr. *Is this what your speedometer read during the whole trip?*—probably not, if you stopped for food and gas. Maybe your trip looked like the diagram:



In this case, your velocity (*i.e.*, your speedometer reading) was not constant. What AVERAGE-velocity measures is "at what constant-velocity you would have to travel to make the same trip in the same time". *[On a graph of position-vs.-time, it is* **"the slope of the line joining the starting-point and the final-point"***.]*

Note carefully, that **AVERAGE-velocity depends on the time-interval under consideration.** Contrast this with velocity, which depends only at the instant under consideration.

The failure to distinguish velocity from average-velocity is a common source of error for many PHY 211 students. Get it straight now.

The **acceleration** of an object is its **"rate of change of velocity with time"**. The acceleration is also a directional-quantity. For one-dimensional motion, the acceleration could be positive, negative, or zero. *[On a graph of* **velocity**-*vs.-time, acceleration is* **"the slope of the curve"**. *We will study this interpretation more carefully when we study calculus.]* Acceleration is a measure of "how fast your velocity is changing, and in what direction".

By contrast, the **AVERAGE-accelaration** of an object **OVER AN INTERVAL** is a "smoothed" measure of "how fast your velocity was changing during some period of time".

The Experiment and The Analysis

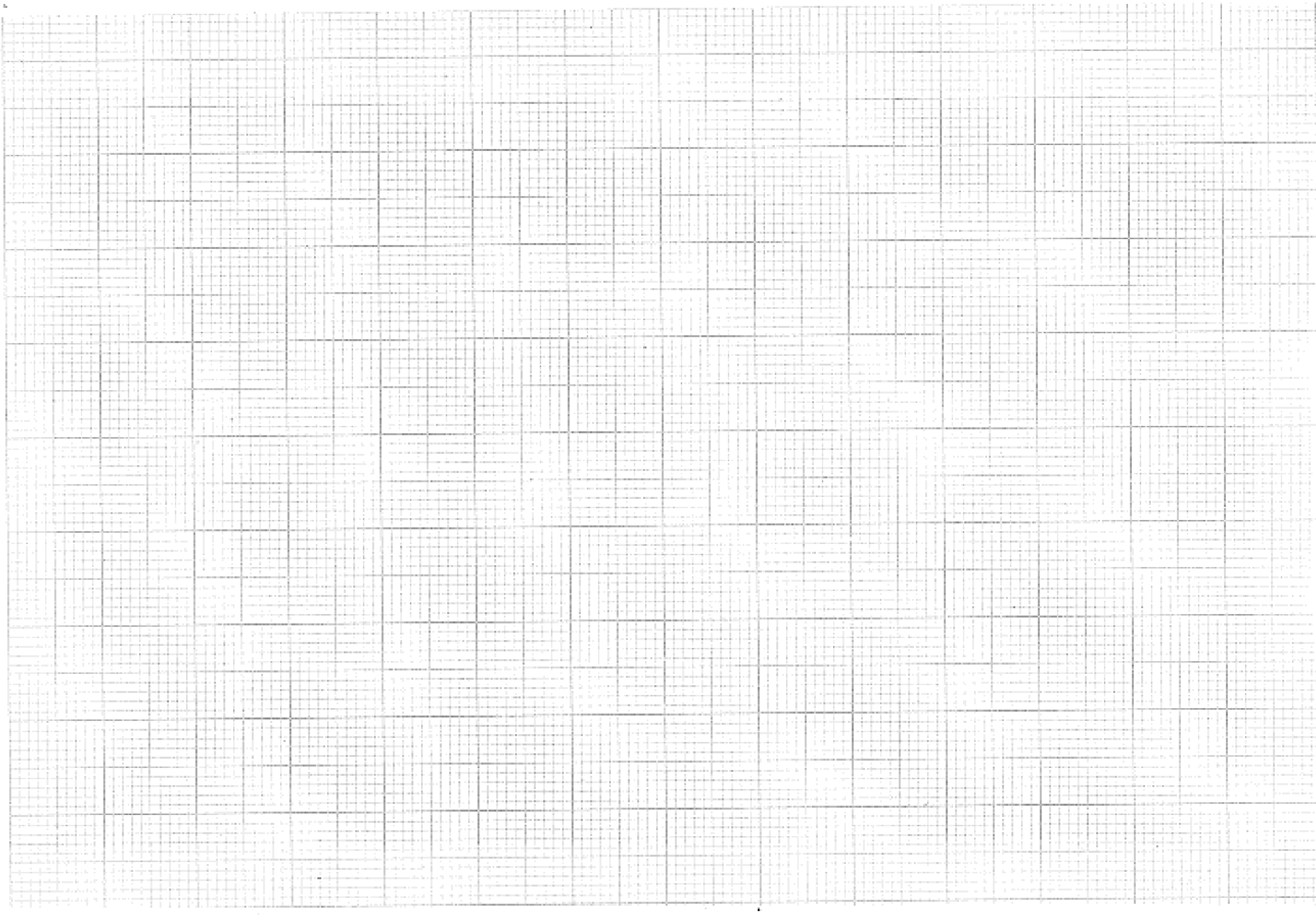
Construct the following table.

time t (s)	displacement x (m)	average velocity v_{avg} (m/s)	average acceleration a_{avg} (m/s^2)
0.00			
0.25			
0.50			
0.75			
1.00			
1.25			
1.50			
1.75			
2.00			
2.25			
2.50			
2.75			
3.00			
3.25			
3.50			
3.75			
4.00			

We want to show that the acceleration is constant. We will do this by showing that its average-acceleration is constant.

[•] Collect the data. Complete the column of positions.

[•] Plot the data on a graph.



From the raw data of "displacement-vs.-time", we need to calculate the "average velocity during each interval". Now,

$$\text{average velocity } v_{avg} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_{end} - x_{start}}{t_{end} - t_{start}}$$

So, for example, during the time-interval between $t = 2.00$ s and $t = 2.50$ s, if the initial position was $x = 1.00$ m and the final position was $x = 1.40$ m, we have $v_{avg} = \frac{1.40-1.00}{2.50-2.00} = 0.80$ m/s. We place this average in the middle of the time-interval, *i.e.*, at $t = 3$ s. Thus, $v_{avg}(3) = 3.5$ m/s

☐ Complete the column of average-velocities.

Now, given the "average-velocities vs. time", we can crudely calculate the "average-acceleration" during each interval. So,

$$\text{average acceleration } = a_{avg} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{v_{end} - v_{start}}{t_{end} - t_{start}}$$

☐ Complete the column of average-accelerations.

You should find that a_{avg} is approximately constant, with possibly a sizeable experimental uncertainty, however. This is to be expected since we took so few data points. Nevertheless, it has served our purpose of crudely verifying (part of) the law of falling bodies.

Of course, the other part of the law of falling bodies is that "all" bodies (*i.e.*, without regard to composition or mass) fall at that same constant acceleration.

So, we should really repeat the experiment with a different mass. Instead, let's view a videotape of an experiment done on the moon by one of our astronauts.

[•] What was observed?

The general law of motion for falling bodies near the earth is often written:

$$x = x_0 + v_0 t - \frac{1}{2} g t^2,$$

where x_0 is a constant (the initial height), v_0 is another constant (the initial velocity), and g is another constant (the acceleration due to gravity, which has a value of 9.8 m/s^2). The minus-sign appears because one usually calls "upward" (*i.e.*, away from the center of the earth) the "positive" direction, and because things are accelerated by the earth's gravity "downward" (*i.e.*, towards the center of the earth).

The general law of motion for falling bodies has the form of a quadratic function "$x = at^2 + bt + c$". On a position-vs.-time graph, this is a parabola.

Solving Equations

Suppose a body falls into a pit from rest (*i.e.*, $v_0 = 0$) at ground level (*i.e.*, $x_0 = 0$). Thus, the law of motion looks like

$$x = -\frac{1}{2} g t^2.$$

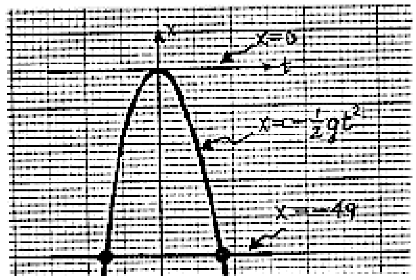
We could ask "When is $x = -49$ m?" In other words, "For what value of t is $x = -49$?"

Algebraically, we are asking "Given $-49 = -\frac{1}{2} g t^2$ with $g = 9.8$, find t." One can easily "solve this equation for t" (*i.e.*, perform algebraic manipulations to both sides of the equation in order to isolate t on one side). Thus,

$$\begin{aligned} -49 &= -\frac{1}{2} g t^2 \\ \frac{2}{g}(49) &= t^2 \\ t &= \pm\sqrt{\frac{2}{g}(49)} \text{ (we will take the + root)} = \sqrt{10} \approx 3.16. \end{aligned}$$

[It is best to **do all of the algebra first**. Plug-in (*i.e.*, do arithmetic) at the last step. Exact answers (*e.g.*, $\sqrt{10}$) are preferred over approximate answers (*e.g.*, 3.16).]

Geometrically, we are asking "Given the graph of our function $x = -\frac{1}{2}gt^2$ and the graph of the line $x = -49$, find t for which the two graphs intersect."



Although the algebraic method and the geometric method are equivalent, the algebraic method will be more useful in performing the calculation. However, in my opinion, the geometric method provides an intuitive view of what one is doing when one is solving an equation.

For example, in this example, one could have been careless algebraically and not realized that there are two possible values of t which satisfies the equation, $\sqrt{10}$ and $-\sqrt{10}$. However, with a geometric picture of the situation, one would see immediately both values.

⊡ Why did we take the positive-root? Could the negative-root have any meaning in this problem or a variation of this problem?

Note that the act of **"finding the roots of an equation"** is essentially the above procedure using $x = 0$.

Systems of Equations

In the previous section, we discussed a special case of a **system of equations** (*i.e.*, a set of equations which must be solved simultaneously). This special case was the following set of equations:

$$\begin{aligned} x &= -\frac{1}{2}gt^2 \\ x &= -49 \end{aligned}$$

The act of **"solving the set of equations simultaneously"** means finding all points that simultaneously lie on both graphs. In other words, it means finding all points-of-intersection between the two graphs.

Consider the following set of equations: $$\begin{aligned} y &= 3x + 5 \\ y &= 2x - 5 \end{aligned}$$

- Plot both (linear) equations on the same graph.

- Geometrically, find the point(s) of intersection, if any.
  Algebraically, solve that system of equations.

One can proceed in many ways. For example, solve one equation for y (it's already done in this example), then plug this expression for y into the other equation. Next, solve the resulting equation (which should not have any more y's in it) for x. Given this value of x, plug into one of your original equations to solve for the value of y. The two values x and y comprise the simultaneous solution to this system of equations.

- Does your algebraic solution for x and y, treated as the coordinates of a point (x, y) on the graph, coincide with the point of intersection obtained with the geometric method?

Consider the following set of equations: $$\begin{aligned} y &= 3x + 5 \\ y &= 3x + 6 \end{aligned}$$

- Plot both (linear) equations on the same graph.

- Geometrically, find the point(s) of intersection, if any.
  Algebraically, solve that system of equations.

Consider the following set of equations: $$\begin{aligned} y &= 3x + 5 \\ -2y &= -6x - 10 \end{aligned}$$

- Plot both (linear) equations on the same graph.

- Geometrically, find the point(s) of intersection, if any.
  Algebraically, solve that system of equations.

Bonus Question (1 pt):

☉ Can this be correct?

Let	x	$=$	y	
	x^2	$=$	xy	Multiply both sides by x
	$x^2 - y^2$	$=$	$xy - y^2$	Subtract y^2 from both sides
	$(x-y)(x+y)$	$=$	$(x-y)y$	Factor
	$x + y$	$=$	y	Cancel out common factor
	$y + y$	$=$	y	Use $x = y$
	$2y$	$=$	y	Combine terms
	2	$=$	1	Cancel out common factor

Where is the error?