PHY 209
Space and Time in Elementary Physics

Triangles and Trigonometry

The Triangle

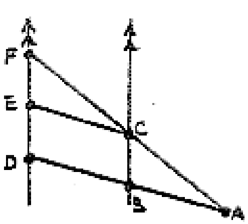
The triangle is the simplest of the polygons. In fact, every polygon can be broken into triangles. (This was one way to measure the area of an irregular polygon from a previous assignment.) Some useful facts that one should know:

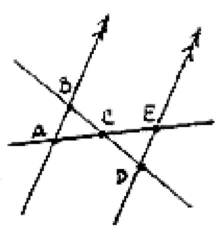
- the area of a triangle is $A = \frac{1}{2}bh = \frac{1}{2}$(base)(height)
- the sum of the interior-angles of a triangle is 180 degrees
- (*the Triangle Inequality*) the sum of the lengths of any two sides is greater than the length of the third
- (*the Law of Sines*) the largest side is opposite the largest angle; the smallest angle is opposite the smallest angle

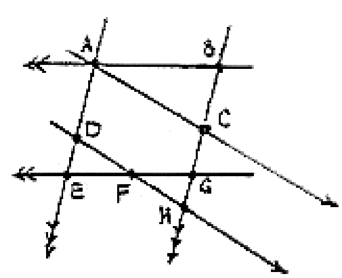
Similar Triangles

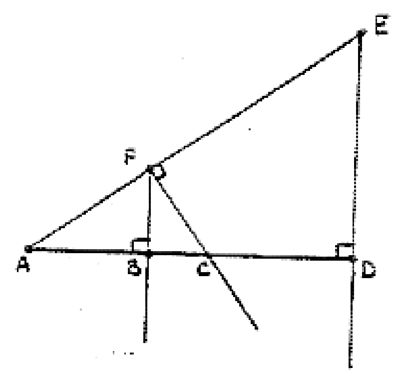
Two triangles are called <u>similar triangles</u> if they have their corresponding angles congruent, or equivalently, they have their corresponding sides in proportion. In other words, two triangles are similar if they are scaled (and possibly rotated and then flipped) versions of each other. As we saw in our measurement of the height of the Carrier Dome, one can exploit this condition can be used for making *indirect* measurements of heights.

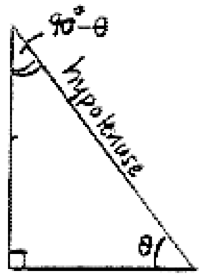
- In each figure, use geometry to identify as many pairs of similar-triangles as possible.











Right Triangles

A triangle with one right-angle is called a **right-triangle**. Of course, since the sum of the interior-angles of any triangle is 180 degrees, a right-triangle has two complementary acute angles. (Recall: an angle is *acute* if it has a measure less than 90 degrees; two angles are *complementary* if the sum of their measures is equal to 90 degrees.)

Some useful facts:

- an **isosceles triangle** (*i.e.*, a triangle with two sides or, equivalently, two angles congruent) can always be broken into two congruent right-triangles
- a rectangle can always be broken into two congruent right-triangles.

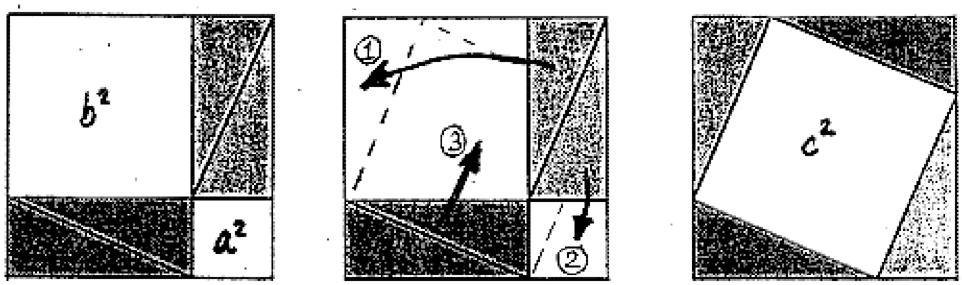
The **hypotenuse** is the side opposite to the right-angle. Since it belongs to the largest angle of the triangle, by the Law of Sines, the hypotenuse is the longest side of a right-triangle. The other sides are called the legs of the right triangle.

One of the most useful properties of a right triangle is the celebrated **Pythagorean Theorem:**

The sum of the squared-lengths of the legs is equal to the squared-length of the hypotenuse.

$$a^2 + b^2 = c^2$$

Below is one of many possible pictorial proofs of this theorem.



Interlude: Geometrical Optics—The Law of Reflection

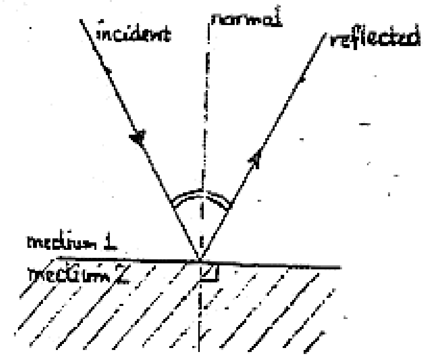
- I have set-up an apparatus that demonstrates some principles of geometric optics. Spend a few minutes to play with the apparatus. (Be careful not point the laser or reflect the laser into anyone's eyes.)

In a uniform medium (*e.g.*, vacuum, air, water), light travels in straight lines. (This is not quite true because we now know that light is bent by gravity. It would be more correct to say that *"light travels in as straight a line as it can"*.)

However, at the surface of contact (called the **interface**) between two mediums, light can be *reflected* and *refracted*. A light ray coming upon the interface is called the **incident ray**. A **reflected light-ray** is light deflected back into the original medium. A **refracted light-ray** is light transmitted into the second medium. Generally, reflected and refracted light-rays are not along the same line as the incident light-ray. In other words, light is generally bent at an interface.

The **Law of Reflection** says:

$$\text{(angle of incidence)} = \text{(angle of reflection)}.$$



These two angles are measured with respect to a certain line, called the <u>**normal**</u> (see diagram). At the point of incidence O, the "normal at O" is the line which is perpendicular to the surface. (Picture a flagpole standing upright at O. For something more graphic, picture "Pinhead" (from some silly horror movie whose name escapes me), whose pins represent the normals at each point on his face.) The <u>**angle of incidence**</u> is the angle between these two lines. Since two non-collinear lines determine a plane, the indicent-light-ray and the normal define a plane (called the <u>**plane of incidence**</u>). The <u>**reflected light-ray**</u> is another light-ray on this same plane. The angle it makes with the normal is given by the Law of Reflection.

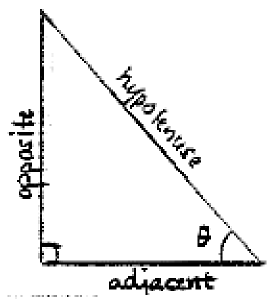
- Select a glass prism or mirror from the apparatus. Trace it out on a sheet of paper. On your paper, choose an incident-ray and, using a protractor, construct the reflected-ray. (If you choose a glass prism, the incident- and reflected-rays of interest will travel within the glass. If you choose a mirror, the incident- and reflected-rays will travel in the air.) When you are done, bring your diagram to the apparatus and we will try to set up your construction and, hopefully, verify the Law of Reflection.

PHY 209
Space and Time in Elementary Physics

Triangles and Trigonometry Part-2

The Trigonometric Functions

Given a right-triangle, one can define functions which relate the acute-angles to certain ratios of the sides of the right-triangle. Taking the θ to be angle as marked in the diagram, we have



$$\sin\theta = \frac{\text{(opposite side)}}{\text{(hypotenuse)}},$$
$$\cos\theta = \frac{\text{(adjacent side)}}{\text{(hypotenuse)}},$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{(opposite side)}}{\text{(adjacent side)}}$$

Of course, since the hypotenuse is the longest side of the right triangle, we must have $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.

- Starting from the Pythagorean Theorem,

$$\text{(opposite side)}^2 + \text{(adjacent side)}^2 = \text{(hypotenuse)}^2,$$

derive the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$.
Ans:

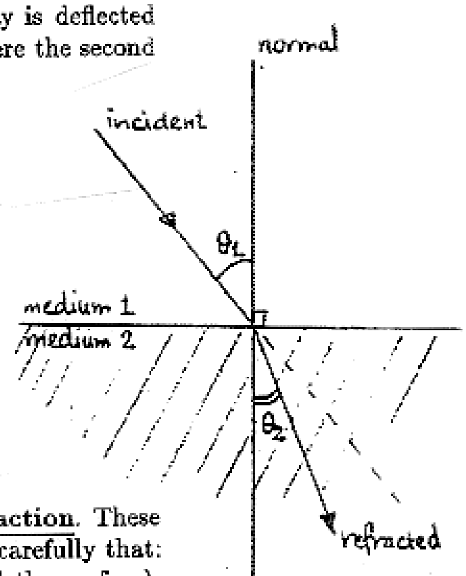
- Using a 30-60-90 right triangle (and possibly a calculator), write down the sin, cos, and tan of the acute angles in terms of ratios of the lengths of the sides of the triangle.
Ans:

For example, in a 45-45-90 triangle, the hypotenuse is $\sqrt{2}$ times longer than both the adjacent and opposite legs. Thus, $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$.

Interlude: Geometrical Optics—The Law of Refraction

- Observe that light rays are bent when light passes through different media, producing optical distortions of reality.

Refracted light gets transmitted from the original medium into the second medium. However, because of a certain optical property of mediums (called the **index of refraction** of the medium), the refracted light-ray is deflected off from the geometrical straight-line-path it would have taken were the second medium absent.



The **Law of Refraction** says:

$$\begin{pmatrix}\text{index of refraction}\\ \text{of medium 1}\end{pmatrix} \sin\theta_1 = \begin{pmatrix}\text{index of refraction}\\ \text{of medium 2}\end{pmatrix} \sin\theta_2,$$

where θ_1 is the **angle of incidence** and θ_2 is the **angle of refraction**. These two angles are also measured with respect to the normal. (Note carefully that: if you use the wrong angle (*e.g.*, the angle between the ray and the surface), you will get the wrong answer.) The refracted light-ray is also on the plane of incidence. Often, the letter "n" is used to denote the index of refraction. Thus, the above is often written

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Physically, the index of refraction is a ratio:

$$\begin{pmatrix}\text{index of refraction}\\ \text{of medium 1}\end{pmatrix} = \frac{\text{the speed of light in vacuum}}{\text{the speed of light in medium 1}}.$$

Thus, for a vacuum, its index of refraction $n_{\text{vacuum}} = 1$. And, since nothing travels faster than the speed of light in vacuum, the index of refraction is never less than 1. For air, $n_{\text{air}} \approx 1.00002$ (which can be taken to be 1 for most purposes). For glass, n_{glass} can vary between 1.2 and 1.6, depending on the glass.

- Select a glass prism from the apparatus. Trace it out on a sheet of paper. On your paper, choose an incident-ray (in the air outside the glass) and, using a protractor, measure the angle of incidence. Bring your diagram to the apparatus and we will try to set up your construction. Trace the refracted-ray on your paper. Using a protractor, measure the angle of refraction. Taking n_{air} to be 1, use your angles and the Law of Refraction to estimate the index of refraction for the glass.

Bonus Question (1 pt):

Two telephone poles (of unequal height and separated by a distance l) stand upright on a level surface. I wish to hang a wire from the top of one (call it A), to some point (call it C) on the ground, back to the top of the other (call it B) such that the total length of the wire $AC + CB$ is as short as possible. Locate the point on the ground that will achieve this. (Neglect "bending" effects due to gravity. Treat sections of the wire as straight line segments.)