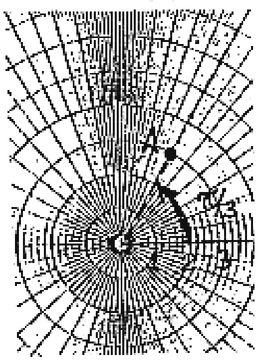
PHY 209
Space and Time in Elementary Physics

Polar Coordinates

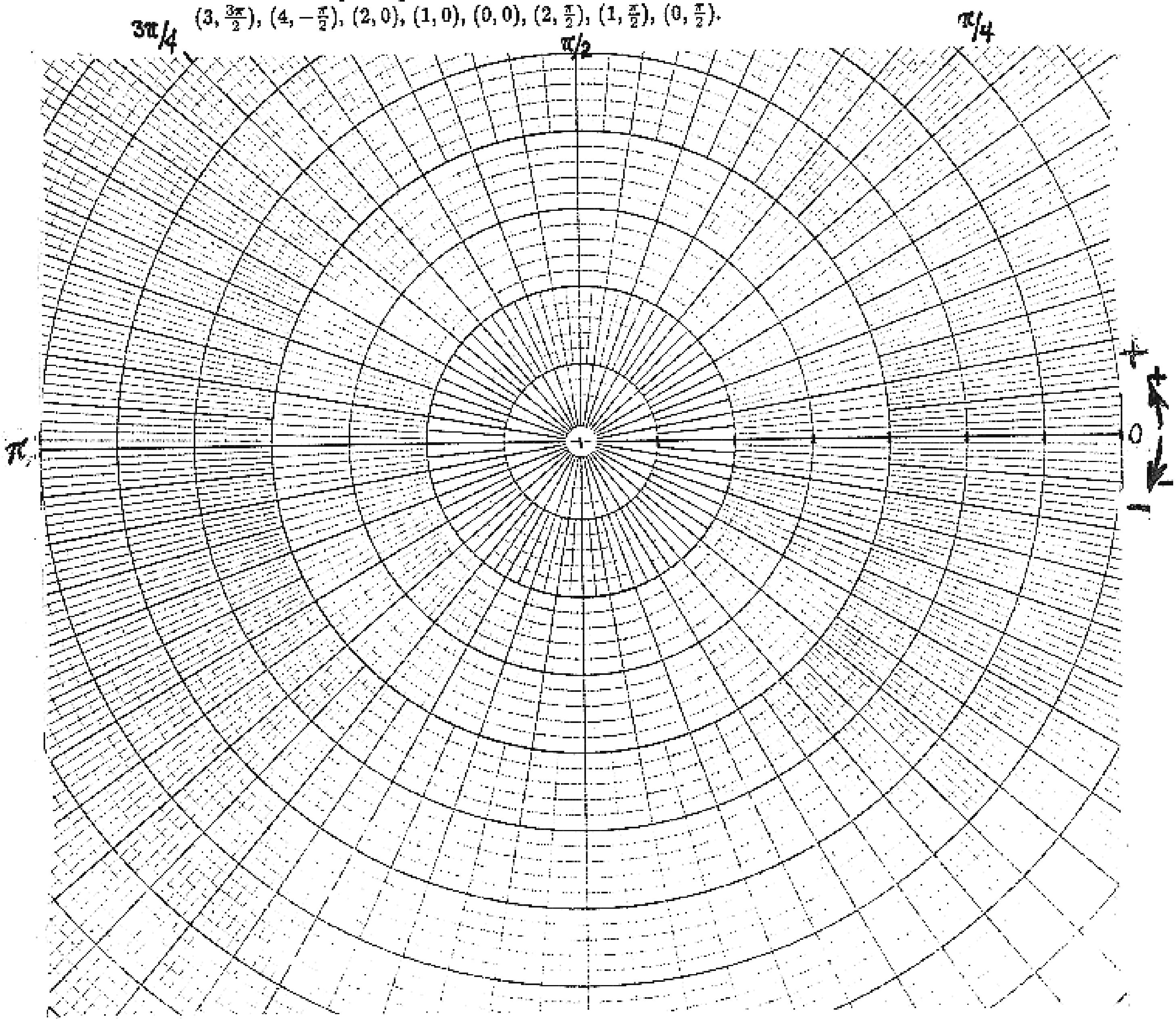


Polar Coordinates

Cartesian coordinates (x, y) provided a way to locate a point on the plane. Here we introduce <u>**Polar coordinates**</u>, another set of coordinates for the plane. Instead of a pair of directed-lines defining the grid, we have a ray (emanating from an origin) and an circle directed *counterclockwise* (centered about that origin).

For instance, consider the grid on the left. The point A is located at $(3, \frac{\pi}{3})$. The first number gives the radius of the circle. The second number gives the angle measured *counterclockwise* from the ray. The order is important because $(\frac{\pi}{3}, 3)$ refers to another point. The "sense" (*i.e.*, that the angle is measured *counterclockwise*) is important because the opposite sense refers to another point.

- On a sheet of polar graph paper, plot the following points $(1, \frac{\pi}{3})$, $(3, \frac{3\pi}{2})$, $(4, -\frac{\pi}{2})$, $(2, 0)$, $(1, 0)$, $(0, 0)$, $(2, \frac{\pi}{2})$, $(1, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$.

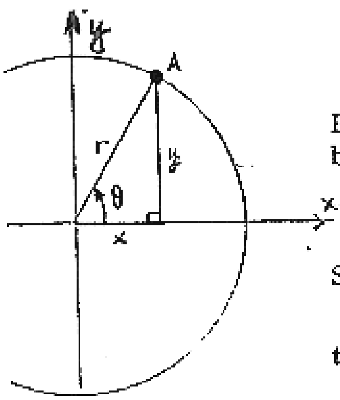


Observe in the above exercise that in the range $(0 \leq r < \infty,\ 0 \leq \theta < 2\pi)$, the origin can be located by more than one pair of polar-coordinates. This is to be contrasted with the Cartesian coordinates where every point has exactly one pair of Cartesian-coordinates. This could be a source of concern for certain applications...but we won't worry about this.

So, (except for the origin) the polar coordinates are just as good as the Cartesian coordinates in locating points on the plane. At this point, a natural question is question is "How are the two descriptions related?" In other words, "how is (r, θ) related to (x, y)"?

Polar Coordinates—the First Quadrant

Consider the following right-triangle with legs aligned with the x- and y-axes. In terms of Cartesian-coordinates, the vertex A has Cartesian-coordinates (x, y). From trigonometry, we know that the angle θ (measured counterclockwise from the x-axis) is given by



$$\tan\theta \equiv \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}, \qquad \text{so, } \theta = \tan^{-1}\frac{y}{x}$$

From the Pythagorean theorem, we know that the hypotenuse (call it r) is given by

$$r = \text{hypotenuse} = \sqrt{(\text{adjacent side})^2 + (\text{opposite side})^2} = \sqrt{x^2 + y^2}.$$

So, in terms of Polar-coordinates, the vertex A has Polar-coordinates (r, θ).

Suppose instead we started with (r, θ). We again know from trigonometry that

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r}.$$

Solving for x and for y, we have immediately

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Polar Coordinates—Outside the First Quadrant

The previous section is fine and good for angles $0 \leq \theta \leq \frac{\pi}{2}$, that is, for points in the "first quadrant". **For angles outside the first quadrant, one must be careful.** Knowing $\tan\theta$ is not the same as knowing θ.

- Let $r = 1$. Let $\tan\theta = 1$. Note: there are two points described by these conditions.

- Given $r = 1$ and $\theta = 45°$, find (x, y).
  Calculate $\tan\theta$ by using $\tan\theta = \frac{y}{x}$.
  Ans:

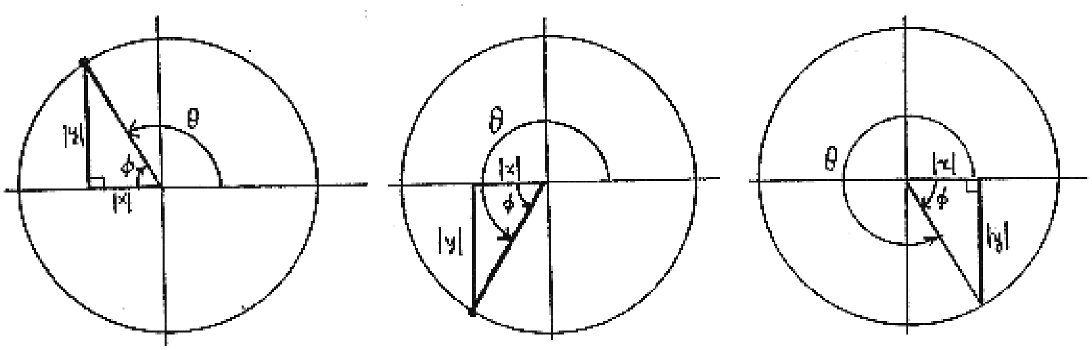
- Given $r = 1$ and $\theta = 225°$, find (x, y).
  Calculate $\tan\theta$ by using $\tan\theta = \frac{y}{x}$.
  Ans:

- How are these two points related to each other?
  Ans:

- Let $r = 1$. Let $\tan\theta = -1$. There are two points (not just one!) described by these conditions. Find these two points. Give the (r, θ)-coordinates and the (x, y)-coordinates of each.

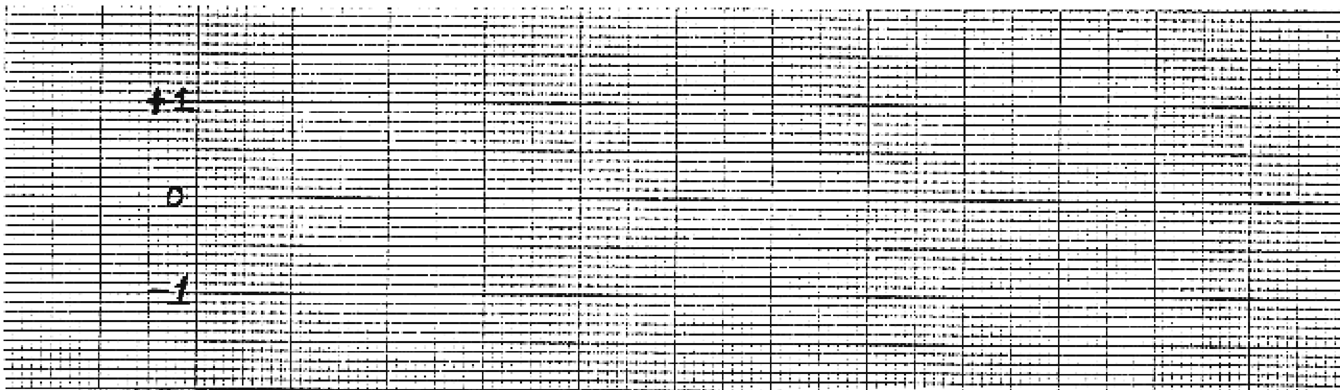
- Looking at these two examples, given r and $\tan\theta$, one needs some extra information to locate just a single point. Formulate a good rule for calculating θ given r, $\tan\theta$, and the sign of x.

- Observe the right-triangles with base $|x|$ and height $|y|$ associated with the other quadrants. We will call ϕ the angle at the origin inside the triangle. Relate θ to ϕ in each quadrant.



Periodic Functions

- Plot on the same (rectangular) graph the functions $\sin\theta$, $\cos\theta$, $\tan\theta$ These from $-2\pi < \theta < 2\pi$.



functions are examples of *periodic functions.* In general, a periodic function is a function $F(\theta)$ that has the property that there exists some number T (called the period) such that $F(\theta + T) = F(\theta)$, for all θ. In other words, a periodic function is *"a function which repeats itself after one period (or cycle)"*. Usually, one takes the period to be the smallest such value of T.

- What is the smallest period of each of $\sin\theta$, $\cos\theta$, $\tan\theta$? *(Hint: Look at your plots above.)*

The Oscilloscope

The oscilloscope is a useful device for studying electrical signals that are periodic in time. Steady signals called simple-harmonic oscillations have the form

$$y = A\sin(\omega t),$$

where the displacement y runs on the vertical scale and time t runs on the horizontal scale.

Amplitude: A is called the amplitude, which measures the maximum displacement of the oscillations (measured in whatever units refer to the oscillator, *e.g.*, meters m for a mass-on-a-spring or volts V for an electrical signal).

Angular Frequency: ω is called the angular-frequency, which measures the rate of the oscillation (measured in radians-per-second rad/s).

Frequency: The angular frequency is related to a more-familiar quantity, the number-frequency, or simply frequency, f (measured in cycles-per-second cycles/s, or Hertz Hz, or simply inverse-seconds s^{-1}), by $\omega = 2\pi f$.

Period: The period T is the length of time it takes for the oscillation to repeat itself once. It is the reciprocal of the number-frequency.

Ans:

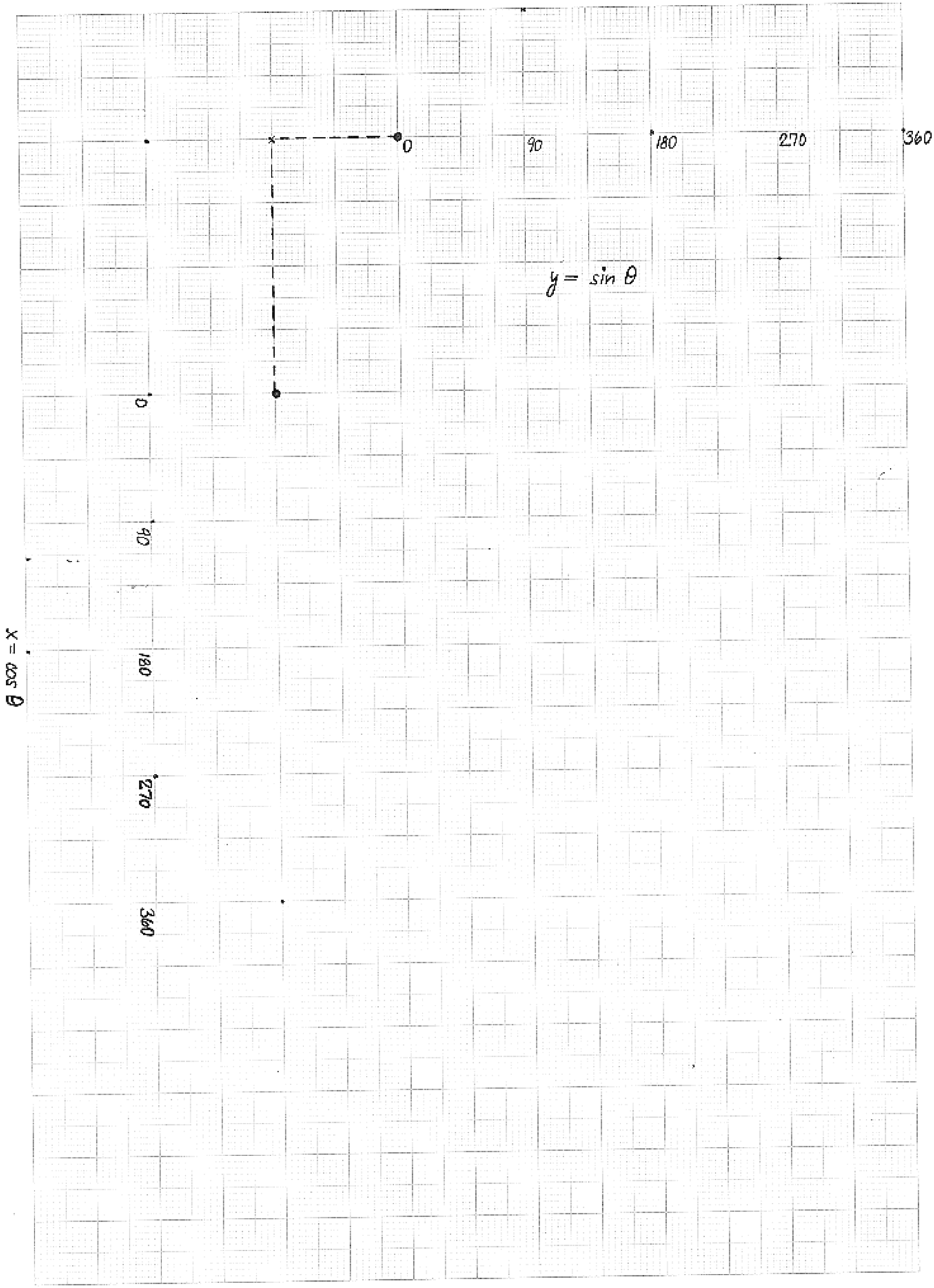
Let $\tan\theta = 1$. Note: there are two points (not just one!) described by these conditions. Your calculator will return an answer of $\theta = 45°$ for $\tan^{-1} 1$. However, what does your calculator return for

- Plot a signal on the graph provided. Label the grid (*e.g.*, what is the scale?) Determine the amplitude, period, and [number-]frequency for each.

- Sketch a plot of what you would expect to see if you
  - doubled the frequency of the oscillation
  - doubled the period of the oscillation.

- Observe two signals (with an oscilloscope and with an audio speaker) with the same amplitude and frequency but with different wave-shapes.

Bonus Question (1 pt):
What does $F(\theta) = \sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \frac{1}{7}\sin 7\theta + \ldots$ produce?


0
90
180
270
360
y = sin θ
0
90
180
270
360
x = cos θ