PHY 209
Space and Time in Elementary Physics

Vectors—Part II

In the previous hand-out, we introduced the vector. We discussed (in a purely mathematical discussion [no physics yet!])

scaling

i.e., given a vector $\vec{A}$, what is the **vector** $k\vec{A}$?

vector-sum

i.e., given two vectors $\vec{A}$ and $\vec{B}$, what is the **vector** $\vec{A}+\vec{B}$ using the "parallelogram law" or the "tip-to-tail method"

Here, we continue our discussion of vectors.

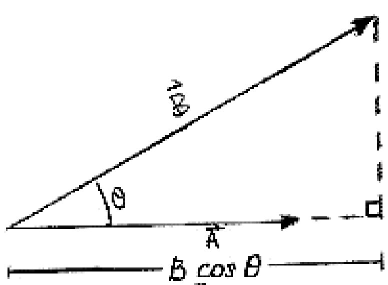
(Euclidean) Dot-Product

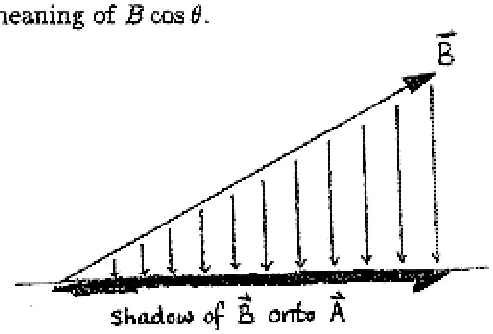
Given two vectors $\vec{A}$ and $\vec{B}$, their dot-product $\vec{A}\cdot\vec{B}$ is a multiplication rule which returns a **scalar** quantity (*i.e.*, a number, essentially). That rule is

$$\vec{A}\cdot\vec{B} = \left\|\vec{A}\right\| \, \left\|\vec{B}\right\| \cos\theta_{\text{between } \vec{A} \text{ and } \vec{B}}.$$

(Often, one writes $\vec{A}\cdot\vec{B} = AB\cos\theta$, where A (without its arrowhead) refers to the magnitude of the vector $\vec{A}$.)

Why does a $\cos\theta$ appear? Observe the geometrical meaning of $B\cos\theta$.



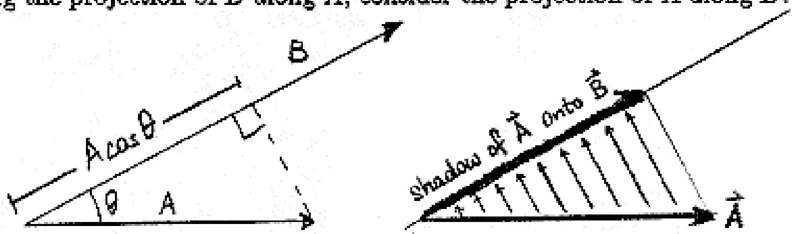


$B\cos\theta$ is the **projection** of $\vec{B}$ along the direction of $\vec{A}$. (You can think of $\vec{A}$ as the ground. Then, with the sun directly overhead, $B\cos\theta$ is the *signed*-length of the shadow. You get a positive-sign if the shadow's arrow points in the same direction as $\vec{A}$. You get a negative-sign for the opposite direction.)

This makes sense since $\cos\theta > 0$ for $|\theta| < 90°$, i.e., $\cos\theta > 0$ in the first and fourth quadrants.

So, in some sense, the dot-product is a measure of the "overlap" of two vectors. It measures "how large a shadow one vector casts on the other, appropriately scaled by the magnitudes of the vectors".

Observe that $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ (*i.e.*, the order doesn't matter). Instead of considering the projection of $\vec{B}$ along $\vec{A}$, consider the projection of $\vec{A}$ along $\vec{B}$.

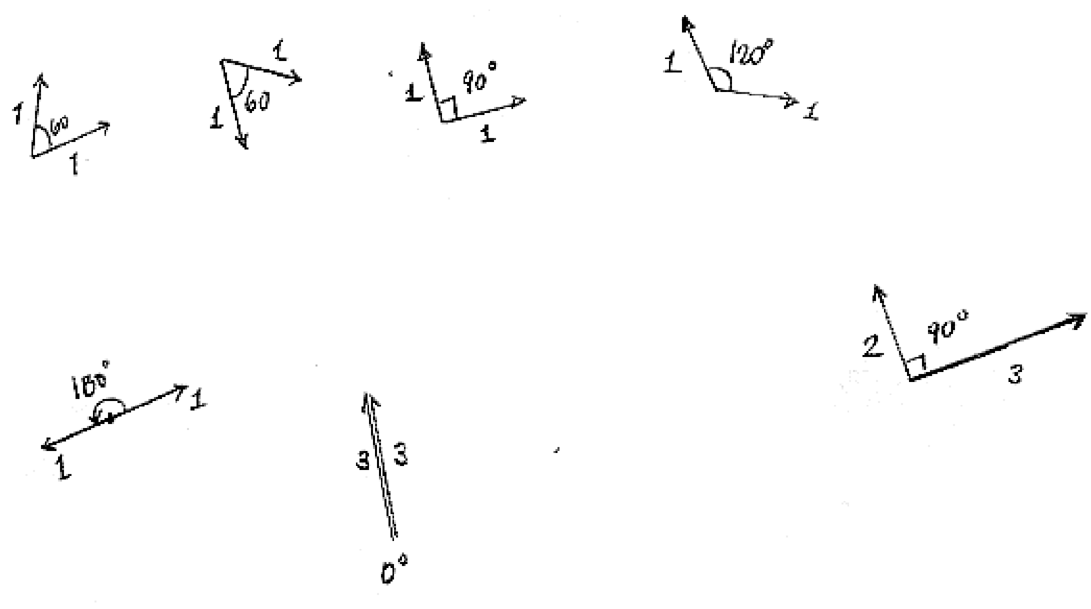


As we said before, the projection of $\vec{B}$ along $\vec{A}$ is $B\cos\theta$. Now, the projection of $\vec{A}$ along $\vec{B}$ is $A\cos\theta$. Of course, generally, $A\cos\theta \neq B\cos\theta$. What *is* generally true is that

$$\vec{A}\cdot\vec{B} = A(B\cos\theta) = AB\cos\theta = BA\cos\theta = B(A\cos\theta) = \vec{B}\cdot\vec{A},$$

regardless of which vector is used to calculate the projections.

- Calculate $\vec{A}\cdot\vec{B}$ for each pair of vectors below. I have given you the angle θ and the magnitudes of the vectors (in meters) in each case. Include in your answer the appropriate units for the dot-product! (Don't be careless. Think about what those units would be).



Here are more useful facts

- $\vec{A} \cdot \vec{A} = AA\cos 0 = A^2$. In other words, the magnitude-of-$\vec{A}$

$$\left\|\vec{A}\right\| = \sqrt{\vec{A} \cdot \vec{A}}$$

- If $\vec{A} \cdot \vec{B} = AB$, (*i.e.*, if $\cos\theta = 1$,), then $\vec{A}$ and $\vec{B}$ are **parallel**. (since $\theta = 0°$)

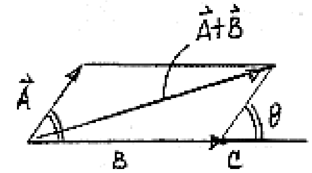
- If $\vec{A} \cdot \vec{B} = -AB$, (*i.e.*, if $\cos\theta = -1$,) then $\vec{A}$ and $\vec{B}$ are **anti-parallel** (*i.e.*, along the same line and pointing in the *opposite* direction). (since $\theta = 180°$)

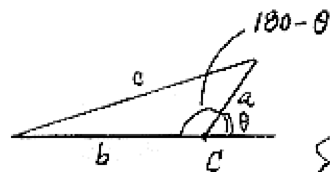
- If $\vec{A} \cdot \vec{B} = 0$, (*i.e.*, if $\cos\theta = 0$,) then $\vec{A}$ and $\vec{B}$ are **perpendicular** or **orthogonal**. (since $\theta = 90°$)

Using the above facts, let us calculate the square of "the magnitude of the vector-sum $\vec{A} + \vec{B}$".

$$\begin{aligned}
\left\|\vec{A}+\vec{B}\right\|^2 &= (\vec{A}+\vec{B}) \cdot (\vec{A}+\vec{B}) \\
&\overset{\text{FOIL}}{=} (\vec{A} \cdot \vec{A}) + (\vec{A} \cdot \vec{B}) + (\vec{B} \cdot \vec{A}) + (\vec{B} \cdot \vec{B}) \\
&= \left\|\vec{A}\right\|^2 + 2(\vec{A} \cdot \vec{B}) + \left\|\vec{B}\right\|^2
\end{aligned}$$

This is essentially the **Law of Cosines** $c^2 = a^2 + b^2 - 2ab\cos C$. From the above calculation, we get the minus-sign in the Law of Cosines when we correctly associate the symbols in the two formulas.





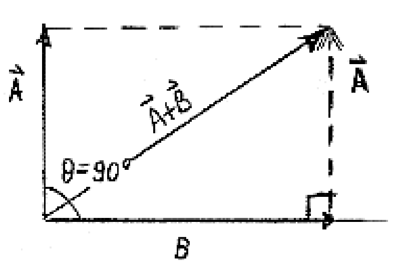
$\cos C = \cos(180-\theta) = -\cos\theta$
so, $2(\vec{A}\cdot\vec{B}) = 2ab\cos\theta$
$= -2ab\cos C$

Now, consider two cases:

- If $\vec{A} \cdot \vec{B} = 0$, *i.e.*, if $\cos\theta = 0$, *i.e.*, when the vectors are perpendicular, then

$$\left\|\vec{A}+\vec{B}\right\|^2 = \left\|\vec{A}\right\|^2 + \left\|\vec{B}\right\|^2$$

but this is nothing but the Pythagorean Theorem, where $\vec{A}$ and $\vec{B}$ are the legs of a right-triangle (*i.e.*, the angle θ between $\vec{A}$ and $\vec{B}$ is $90°$, which is supplementary to the right-angle inside the triangle) and $\vec{A} + \vec{B}$ is the hypotenuse.



• If $\vec{A} \cdot \vec{B} = AB$, *i.e.*, if $\cos\theta = 1$, *i.e.*, when the vectors are parallel, then

$$\begin{aligned} \left\|\vec{A}+\vec{B}\right\|^2 &= \left\|\vec{A}\right\|^2 + 2(AB) + \left\|\vec{B}\right\|^2 \\ &= A^2 + 2(AB) + B^2 \\ &= (A+B)(A+B) \\ &= (A+B)^2 \\ &= \left(\left\|\vec{A}\right\| + \left\|\vec{B}\right\|\right)^2 \end{aligned}$$

So, carefully taking the square-root of both sides,

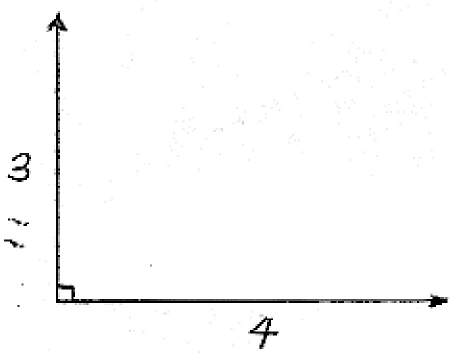
$$\left\|\vec{A}+\vec{B}\right\| = \left\|\vec{A}\right\| + \left\|\vec{B}\right\|.$$

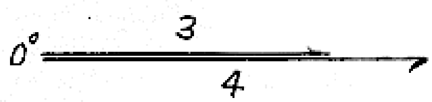
Thus, ***only when the vectors are parallel*** is the "magnitude of the vector-sum" equal to the "sum of the vector-magnitudes".

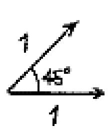
Failure to recognize this fact is probably one of the biggest mistakes made by beginning physics students. GET IT RIGHT!

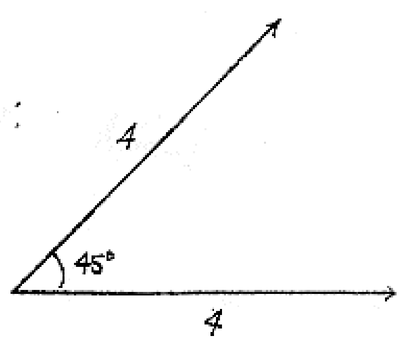
• Calculate $\left\|\vec{A}+\vec{B}\right\|$ for each pair of vectors below.

Hint: Calculate $\left\|\vec{A}+\vec{B}\right\|^2$ using $(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$









As a check for yourself, draw in $\vec{A}+\vec{B}$ and measure its magnitude.