PHY 209
Space and Time in Elementary Physics

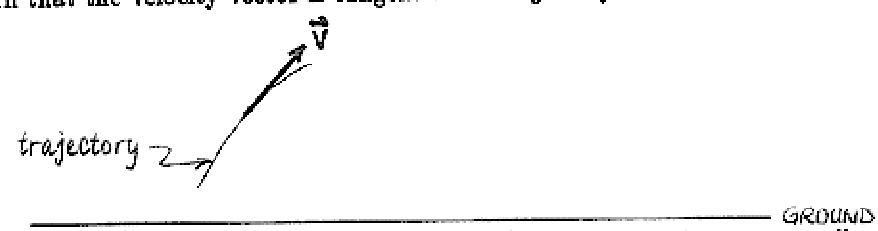
Vectors—Part III (components)

Up to now, we have been emphasizing the geometric approach to vectors. This geometric approach is very concise. *A picture is worth a thousand words (and numbers)!* In describing a physical system mathematically, it is often useful to use a concise mathematical object which neatly encapsulates the physical principles. In some sense, the geometrical picture of a vector as an arrow vividly conveys the notion of a force (loosely speaking, a push or a pull) being applied.

Vector Components

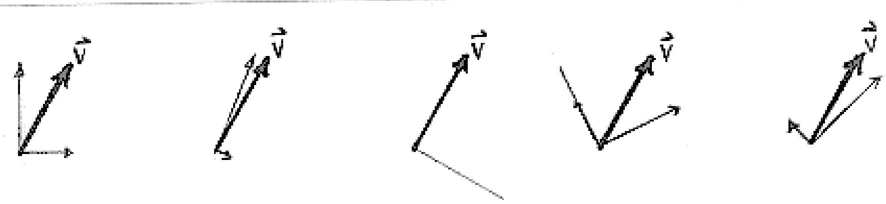
There is another approach to vectors which is very useful when it comes down to making *measurements* of physical quantities.

For example, in discussing the motion of a particle in space, its "instantaneous rate-of-change of position with time" (called its **velocity**) is neatly described by a vector (called, of course, its **velocity vector**). Later, we will learn that the velocity vector is tangent to its trajectory.



Suppose we (on the ground) want to know how fast the particle is traveling parallel to the ground. That is, *"how fast would I have to run on the ground to keep the particle directly overhead as it travels?"* How would we extract that information from the velocity vector?

To answer this, one needs to realize the following:
any vector in two-dimensions can be written as the sum of two mutually perpendicular vectors, i.e., as two vectors, each of which is perpendicular to the other.

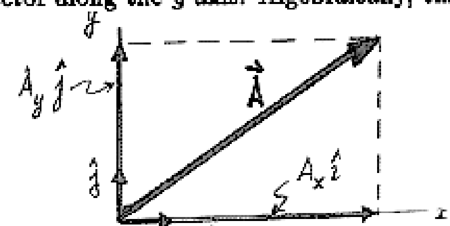


A pictorial way to think of this is that *"any vector in two-dimensions can be thought of as 'the hypotenuse of a right-triangle' "*.

It should be obvious that a two-dimensional vector may be written many different ways as the sum of two mutually perpendicular vectors. Or, in the pictorial way, a two-dimensional vector may be thought of as the hypotenuse of many different right-triangles. **Each way is just as good as any other way** since they describe the *same* vector.

If, however, we are given a rectangular (Cartesian) coordinate system (say, with an x-axis and a y-axis), we are provided a **preferred** set of perpendicular vectors: vectors pointing along the positive directions of the coordinate axes.

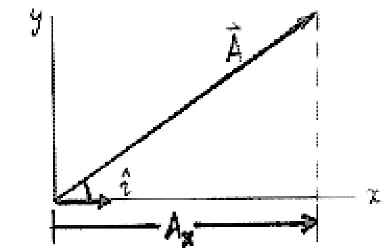
Thus, given some vector $\vec{A}$, we can write $\vec{A}$ as the sum of a vector along the x-axis and a vector along the y-axis. Algebraically, this is expressed as:



$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath},$$

where $A_x\hat{\imath}$ is a vector along the x-axis (called the <u>x-component</u> of vector $\vec{A}$). The symbol $\hat{\imath}$ refers to a *vector* of length 1 pointing along the positive-x direction; this is called the <u>unit-vector in the x-direction</u>. The hat ^ reminds us that this is a unit-vector. (On a sheet of graph-paper, you can think of $\hat{\imath}$ as a "tick-mark" along the x-axis which is one unit from the origin.) The *signed*-number A_x is the scale factor that one must scale the unit-vector $\hat{\imath}$ by, in order to get the x-component of the vector $\vec{A}$.

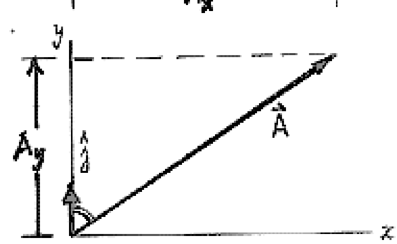
With our discussion of the dot-product above, we can express this idea another way. The "x-component of the vector $\vec{A}$" is a vector pointing in the x-direction whose length is obtained by **projecting** the vector $\vec{A}$ onto the unit-vector in the x-direction. In other words,

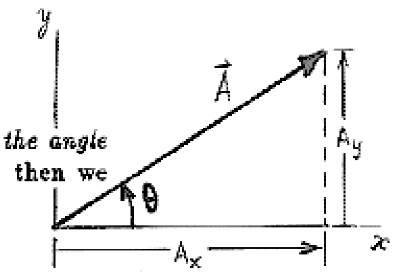


$$A_x = \hat{\imath}\cdot\vec{A} = A\cos\left(\begin{array}{c}\text{angle between}\\ \vec{A}\text{ and }\hat{\imath}\end{array}\right).$$

Similarly,

$$A_y = \hat{\jmath}\cdot\vec{A} = A\cos\left(\begin{array}{c}\text{angle between}\\ \vec{A}\text{ and }\hat{\jmath}\end{array}\right).$$





If we express these formulas in terms of a *single* angle θ, namely, *the angle drawn counterclockwise from the positive-x axis* (polar coordinates!), then we can write

$$A_x = \hat{\imath} \cdot \vec{A} = A\cos\theta \qquad A_y = \hat{\jmath} \cdot \vec{A} = A\sin\theta.$$

I cannot emphasize enough that: *this definition using* $\cos\theta$ *and* $\sin\theta$ *requires that the angle* θ *is the angle drawn counterclockwise from the positive-x axis.* When followed correctly, these formulas already account for minus-signs that arise when the vector is not in the first quadrant. **That's one reason why we studied polar coordinates!**

I EMPHASIZE.
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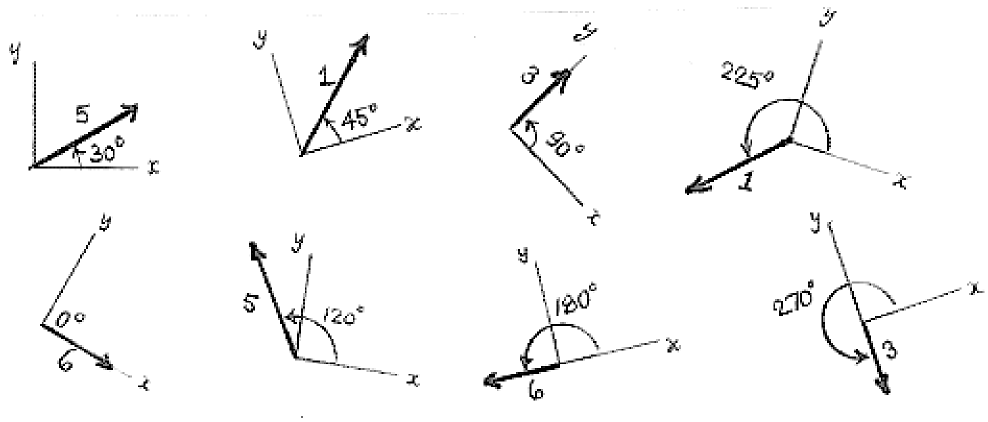
☐ Find the x-components and y-components of each of the vectors below.



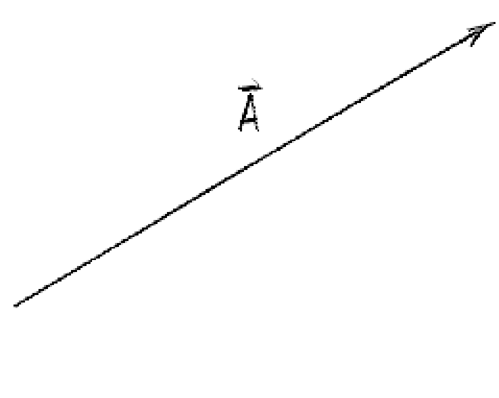
☐ Using the formulas that convert between rectangular and polar coordinates, show that $A = \sqrt{A_x{}^2 + A_y{}^2}$ and that $\theta = \text{arc-tan}\dfrac{A_y}{A_x}$

Observe that if you use a *different set of axes*, you will get *different* values of A_x and of A_y... but, **you will always get the same value when you calculate the magnitude of $\vec{A}$: $\sqrt{A_x^2 + A_y^2}$.**

☐ Take a sheet of translucent graph paper. On the top half of the sheet, draw a set of x- and y-axes. Place the tail of the vector $\vec{A}$ at the origin. Draw in $\vec{A}$. Find the x- and y-components of $\vec{A}$. Calculate $\sqrt{A_x^2 + A_y^2}$.

☐ Repeat this procedure using the bottom half of the graph paper. Still place the tail of the vector at the origin. However, this time, rotate the vector about the origin (so that it makes a differrent angle with the x-axis). Find the x- and y-components of $\vec{A}$. Calculate $\sqrt{A_x^2 + A_y^2}$.

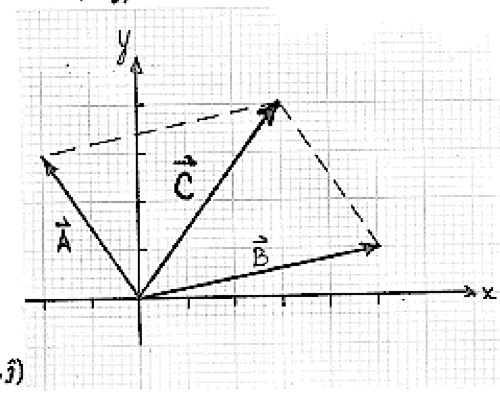
☐ Depending on how carefully you did this, your values for $\sqrt{A_x^2 + A_y^2}$ should agree. As a check, line up the vector along the x-axis, and read off the magnitude of $\vec{A}$.



Vector Addition - componentwise

In terms of components, vector addition can be done algebraically. *Be a good bookkeeper. Keep track of your symbols and "don't add apples and oranges".*

For instance, given the vectors $\vec{A} = -2\hat{\imath} + 3\hat{\jmath}$ and $\vec{B} = 5\hat{\imath} + 1\hat{\jmath}$, let $\vec{C}$ be the vector-sum.



$$\begin{aligned}\vec{C} &= \vec{A} + \vec{B} \\ &= (-2\hat{\imath} + 3\hat{\jmath}) + (5\hat{\imath} + 1\hat{\jmath}) \\ &= (-2+5)\hat{\imath} + (3+1)\hat{\jmath} \\ &= (3)\hat{\imath} + (4)\hat{\jmath} \\ &= 3\hat{\imath} + 4\hat{\jmath}\end{aligned}$$

Generally, given $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$,

$$\begin{aligned}\vec{C} &= \vec{A} + \vec{B} \\ &= (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\ &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath}\end{aligned}$$

so, $C_x = A_x + B_x$ and $C_y = A_y + B_y$. In words, the "x-component of the vector-sum" is equal to the "sum of the x-components", and similarly for the y-components.

It is important not to leave out the unit-vectors $\hat{\imath}$ and $\hat{\jmath}$. If you do, you might end up *"adding apples and oranges" and that's not good bookkeeping!*

The Dot-Product - componentwise

Recall that we defined the dot-product between two vectors $\vec{A}$ and $\vec{B}$ as

$$\vec{A} \cdot \vec{B} = AB\cos\theta,$$

where θ is the angle between the two vectors.

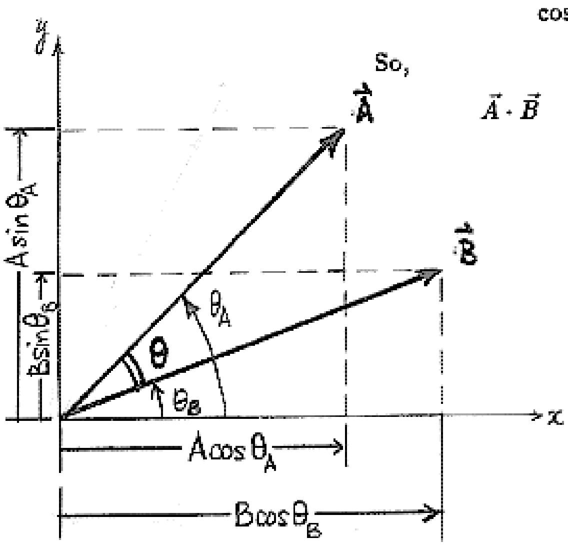
If we know $\vec{A}$ and $\vec{B}$ in terms of their components (relative to a given set of axes), we can write their dot-product as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y,$$

in other words, as *"the sum of the-product-of-the-x-components and the-product-of-the-y-components.*

To see this, we will need to use a trigonometric identity:

$$\cos(\theta_A - \theta_B) = \cos\theta_A \cos\theta_B + \sin\theta_A \sin\theta_B.$$



So,

$$\begin{aligned}\vec{A} \cdot \vec{B} &= AB\cos\theta \\ &\quad \text{note that } \theta_A = \theta_B + \theta \\ &= AB\cos(\theta_A - \theta_B) \\ &= AB(\cos\theta_A \cos\theta_B + \sin\theta_A \sin\theta_B) \\ &= (A\cos\theta_A)(B\cos\theta_B) + (A\sin\theta_A)(B\sin\theta_B) \\ &= A_x B_x + A_y B_y\end{aligned}$$

Observe once again that if you use a *different set of axes*, you will get *different* values of A_x, of A_y, of B_x, and of B_y... but, **you will always get the same value when you calculate the dot-product $A_xB_x + A_yB_y$.**

There is another way to see where this formula comes from.
Let $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$. Using the dot-product,

$$
\begin{aligned}
\vec{A}\cdot\vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath})\cdot(B_x\hat{\imath} + B_y\hat{\jmath}) \\
&\overset{FOIL}{=} (A_x\hat{\imath}\cdot B_x\hat{\imath}) + (A_x\hat{\imath}\cdot B_y\hat{\jmath}) + (A_y\hat{\jmath}\cdot B_x\hat{\imath}) + (A_y\hat{\jmath}\cdot B_y\hat{\jmath}) \\
&\overset{collect}{=} A_xB_x(\hat{\imath}\cdot\hat{\imath}) + A_xB_y(\hat{\imath}\cdot\hat{\jmath}) + A_yB_x(\hat{\jmath}\cdot\hat{\imath}) + A_yB_y(\hat{\jmath}\cdot\hat{\jmath})
\end{aligned}
$$

Now observe that $\hat{\imath}\cdot\hat{\imath} = 1$ and $\hat{\jmath}\cdot\hat{\jmath} = 1$, and $\hat{\imath}\cdot\hat{\jmath} = 0$ and $\hat{\jmath}\cdot\hat{\imath} = 0$

$$
\begin{aligned}
&= A_xB_x(1) + A_xB_y(0) + A_yB_x(0) + A_yB_y(1) \\
&= A_xB_x + A_yB_y
\end{aligned}
$$

☐ For each set, use x- and y-components of $\vec{A}$ and $\vec{B}$ to find their vector-sum and their dot-product.