# PHY 209
# Space and Time in Elementary Physics

## Introduction to Differential Calculus

The word *calculus* means "method of calculating". The word *differential* means "linear approximation"—in other words, "approximation by a line".

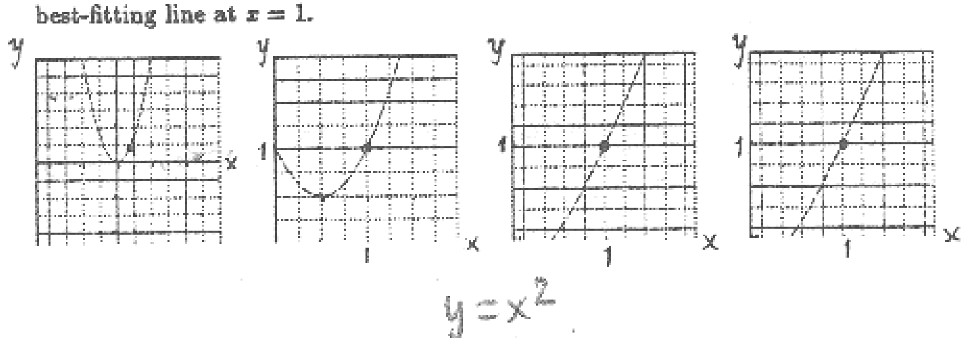
### The Derivative of a Function *at a Point*

On the previous handout, you graphed some functions, zooming-in on a particular point (whose relevance will be revealed in this assignment). Hopefully, you observed that:

> As you zoomed-in to the point of interest, "the graph of the function *near that point*" looked more and more like *"a line through that point"*.

At the point of interest, this line is the *"best-fitting line"* to the graph of the function near that point. In other words, if you are interested in the values of the function near the point of interest, to a good approximation, you can just use the values read-off from this best-fitting line. (If you are interested in points away from the point of interest, this line might no longer be a good approximation.)

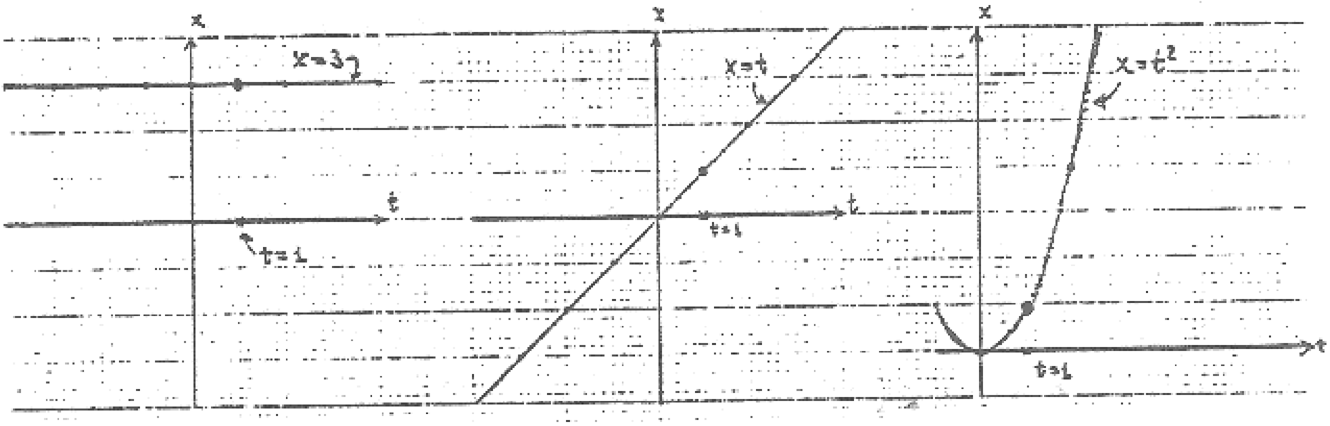
The "slope of this best-fitting line" through the point is the "derivative of the function" at that point. For example, we will learn that "the derivative of $y = x^2$ at $x = 1$" is equal to 2, which we can be seen to be the slope of this best-fitting line at $x = 1$.




Although this method to determine the derivative is intuitive, it is cumbersome to carry out this process. Below we will introduce another method—one which will lead us to quickly calculate the derivative of practically any function. This does not mean that we should abandon this graphical way of thinking about the derivative. Instead, we should retain it, and used it to complement this new method below.

## Time-Out: Motion along a line (revisited)

Below are three sets of data measuring position $x$ vs. time $t$.




Recall that

$$\text{"average velocity (during a time-interval)"} = \frac{\text{"distance traveled"}}{\text{"time elapsed"}}$$

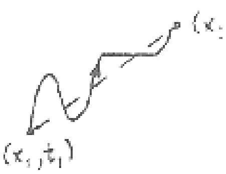
$$= \frac{\Delta x}{\Delta t}$$

$$= \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$= \frac{x(t_1 + \Delta t) - x(t_1)}{(t_1 + \Delta t) - t_1}, \text{ using } \Delta t = t_2 - t_1.$$

$$= \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t}$$

Recall what the *average velocity* means... At time $t_1$, I begin at position $x_1$, then travel somehow [maybe I stop for a second, maybe I turn around then continue forward, maybe I speed up from rest, *etc.*], then, at time $t_2$, I am at position $x_2$.




The *average velocity* is the *constant (steady)* velocity I could have traveled with (instead of the way I actually traveled) to get from position $x_1$ at time $t_1$ to position $x_2$ at time $t_2$. Note, that the average velocity is nothing more than the slope of the chord (line) joining $(x_1, t_1)$ and $(x_2, t_2)$.

- For each of the three graphs above, calculate the average velocity about the time $t_1 = 1$ using each of the five intervals $\Delta t \in \{-2, -1, 0, 1, 2\}$ (in other words, using each of the five other-points $t_2 \in \{-1, 0, 1, 2, 3\}$)

Time-In

Note that, for a linear function, one can choose *any* other point because the slope of a line is constant.

For general functions (*i.e.*, not necessarily linear functions), however, we cannot choose any point $(x_2, t_2)$ to calculate the slope at point $(x_1, t_1)$ since the slope $m$, as computed above, depends on the point $(x_2, t_2)$ we choose.

However, notice that as $\Delta t$ get smaller (*i.e.*, as we choose $t_2$'s that are successively closer to $t_1$), these chords (lines) approach a single line, which is tangent to the curve. This line is the best-approximating line to the curve at $t_1$. The slope of this line is "the slope of the curve at $t_1$"—in other words, the **derivative** of the function at this point.

In symbols,

$$\left.\frac{dx}{dt}\right|_{t=t_1} = \textbf{derivative of } x(t) \text{ at } t_1 = \lim_{\Delta t \to 0} \left.\frac{\Delta x}{\Delta t}\right|_{t=t_1} = \lim_{\Delta t \to 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t}$$

**Time-Out: Motion along a line (revisited)**

When applied to the position $x$ vs. time $t$ graph, the average-velocity for successively smaller time-intervals $\Delta t$ about $t_1$ approaches a quantity called the **instantaneous velocity** (or simply **velocity**) at time $t_1$. Physically, the instantaneous velocity is the "reading of your speedometer"; it is "how fast you are going at that instant".

**Time-In**

**The Derivative of a Function**

In the previous discussion, we were concerned with the slope of the curve at a point. Here, we are interested in the following problem:

> Given a function $x(t)$, find the function $x'(t)$ such that
> for each $a$, this function evaluated at $t = a$, namely $x'(a)$,
> is the slope of the curve $x(t)$ at the point $t = a$.

In other words, given the function $x(t)$, we want the formula for its slope, given by the function $x'(t)$. **Often one uses the notation $\frac{dx}{dt}$ for $x'(t)$.**

This is easy now. We just replace $t_1$ by an arbitrary point $t$ in the above formula. So,

$$\frac{dx}{dt} = \textbf{derivative of } x(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Let's apply see how to apply this formula to various functions.

Let $x(t) = 0.3$.

$$\begin{aligned}
\frac{dx}{dt} &= \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[0.3] - [0.3]}{\Delta t} \\
&= \lim_{\Delta t \to 0} 0 \\
&= 0
\end{aligned}$$

Let $x(t) = t$.

$$\begin{aligned}
\frac{dx}{dt} &= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[t+\Delta t] - [t]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\Delta t}{\Delta t} \\
&= \lim_{\Delta t \to 0} 1 \\
&= 1
\end{aligned}$$

Let $x(t) = t^2$.

$$\begin{aligned}
\frac{dx}{dt} &= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(t+\Delta t)^2] - [t^2]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[t^2 + 2t\Delta t + (\Delta t)^2] - [t^2]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{2t\Delta t + (\Delta t)^2}{\Delta t} \\
&= \lim_{\Delta t \to 0} 2t + \Delta t \\
&= 2t
\end{aligned}$$

- **Evaluate** the above three derivatives at $t = 1$. Does this agree with the slope of the lines obtained on your zoomed-in graphs of these functions?
- **Use the above method** to find the derivative for $x = t^3$ and for $x = t^4$. Hint: $(t+a)^3 = t^3 + 3at^2 + 3a^2t + a^3$ and $(t+a)^4 = t^4 + 4at^3 + 6a^2t^2 + 4a^3t + a^4$.