# PHY 209
# Space and Time in Elementary Physics

## Introduction to Differential Calculus (continued)

**The Derivative of a Function (continued)**

Just to recall what we are doing...

> Given a function $x(t)$, find the function $\frac{dx}{dt}$ such that
> for each value $a$ for the running variable $t$, this function evaluated
> at $t = a$, namely $\left.\frac{dx}{dt}\right|_{t=a}$,
> is the **slope of the curve** $x(t)$ at the point $t = a$.

That is, given the function $x(t)$, we want **the formula for its slope at each point**, given by another function, called its **derivative**, which is written as $\frac{dx}{dt}$. Some other notations for the derivative are $\frac{d}{dt}x$, $x'(t)$ or $\dot{x}(t)$.

So, we had this formula to define the derivative:

$$\frac{dx}{dt} = \textbf{derivative of } x(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Last time, we used this formula to find the derivatives of $x = C$ a constant, $x = t$, $x = t^2$, $x = t^3$. Here is the general result:

Let $x(t) = t^n$.

$$\begin{aligned}
\frac{dx}{dt} &= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \qquad t^n \\
&= \lim_{\Delta t \to 0} \frac{[(t+\Delta t)^n] - [t^2]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[t^n + nt^{n-1}\Delta t + \cdots] - [t^n]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{nt^{n-1}\Delta t + \cdots}{\Delta t} \\
&= \lim_{\Delta t \to 0} nt^{n-1} + \cdots \\
&= nt^{n-1}
\end{aligned}$$

We used the identity $(t+a)^n = t^n + nt^{n-1}a + \cdots$, where we neglected the $(\cdots)$-term because they are very small when $a$ approaches zero. (We would have seen the same behavior if we looked at the graph zoomed-in to any point. Those $(\cdots)$-terms represent the slight deviations away from being a straight line. As we zoom in, we make "$a$" (*i.e.*, $\delta t$) approach zero. Thus, we end up effectively neglecting terms like $a^n$, $a^{n-1}$, *etc.*.

Thus, we have a general rule:

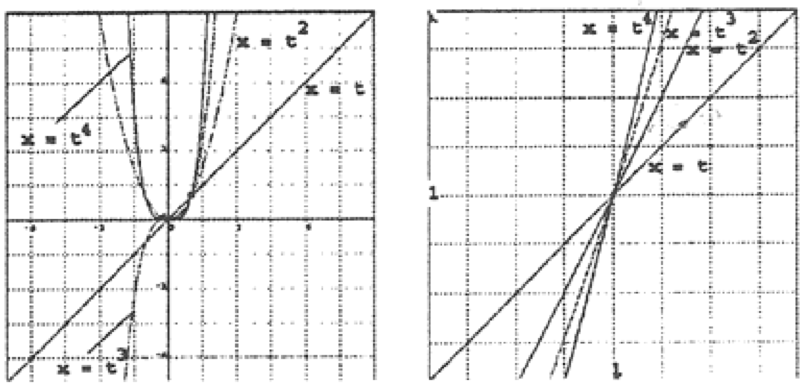
$$\frac{d}{dt}(t^n) = nt^{n-1}$$

If, say, we want to find the derivative of $x = t^n$ *evaluated at* $t = 1$, we substistute "1" for "$t$" everywhere on the right-hand side, and then calculate. Observe that

$$\left.\frac{d}{dt}(t^n)\right|_{t=1} = \left.nt^{n-1}\right|_{t=1} = n$$

since $1^{\text{anything}} = 1$.

This says that "the slope of the graph of $x = t^n$ at $t = 1$" is $n$. Indeed, look at these graphs and their zoom-ins:




$x = t^n$

In elementary physics, one often uses the derivatives of functions to study the behavior of those functions. In other words, one is not only interested in what the values of the functions are. One is also interested in how those values *change* from one point to the next.

**One-Dimensional Motion in a line**

As an example, consider *one-dimensional motion in a line*. An important case is when the position $x$ varies as a function of time $t$ such that

$$x = x_0 + v_0 t + \frac{1}{2} a t^2,$$

where $x_0$, $v_0$, and $a$ are constants.

$x_0$ is the **initial position** of the particle (where the particle was as time $t = 0$). $v_0$ is the **initial velocity** of the particle (what the particle's instantaneous speed at time $t = 0$). $a$ is the **acceleration** of the particle (what the rate-of-change of the speed was at time $t = 0$).

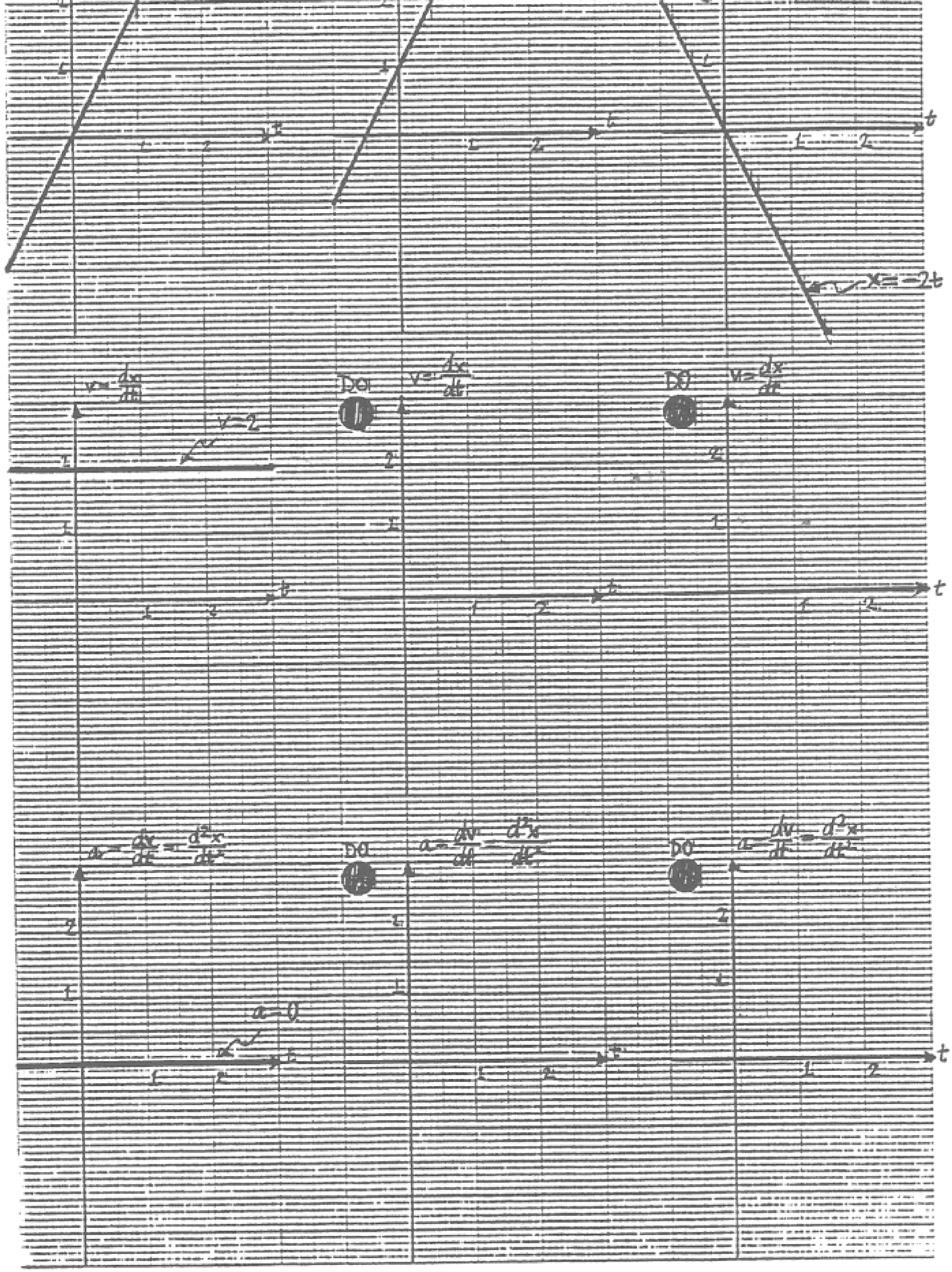
Observe that the **instantaneous velocity at any time** $t$ is given by

$$v = \frac{dx}{dt} = v_0 + at$$

and that the **instantaneous acceleration at any time** $t$ is given by

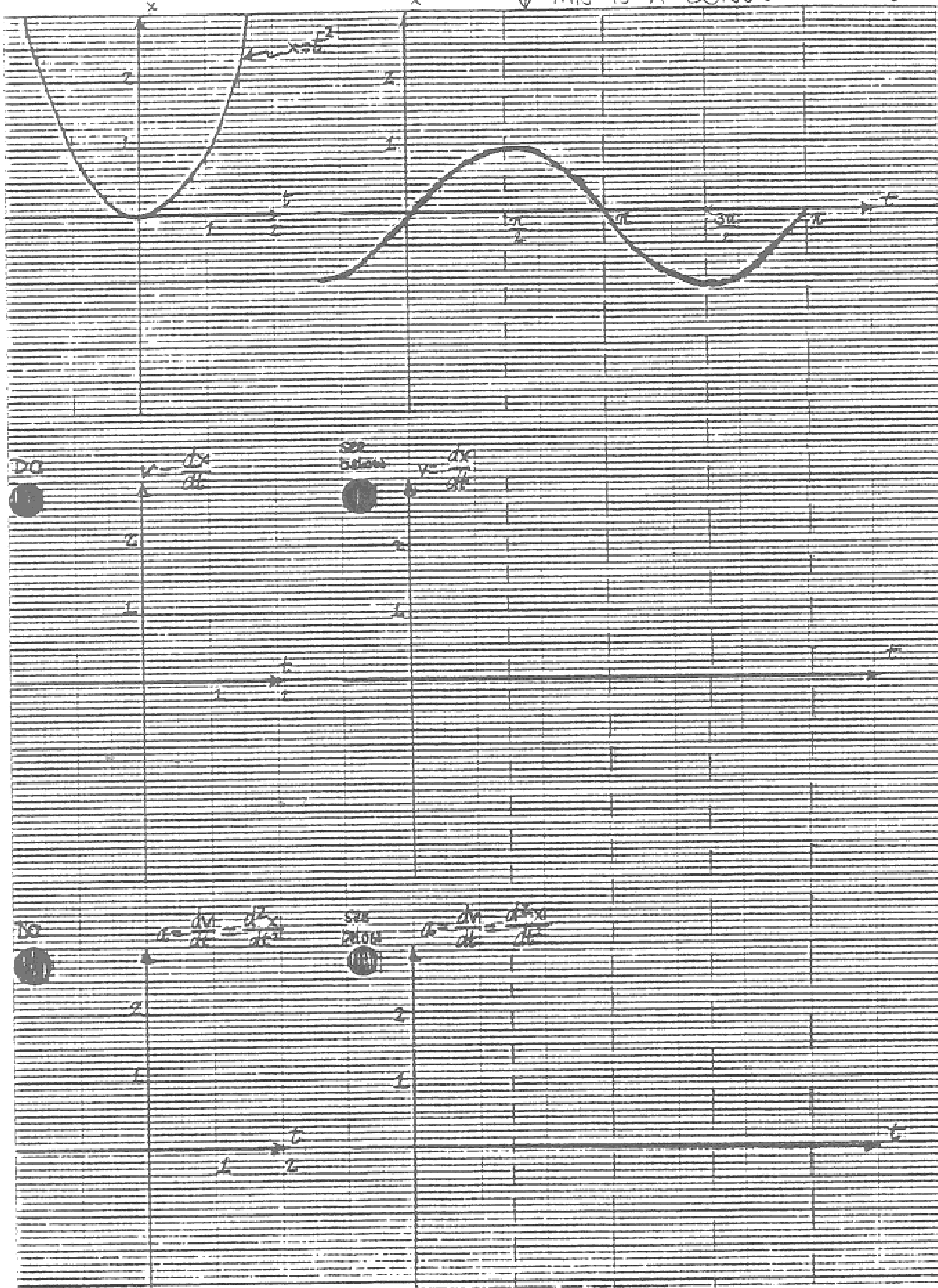
$$a = \frac{dv}{dt} = a.$$

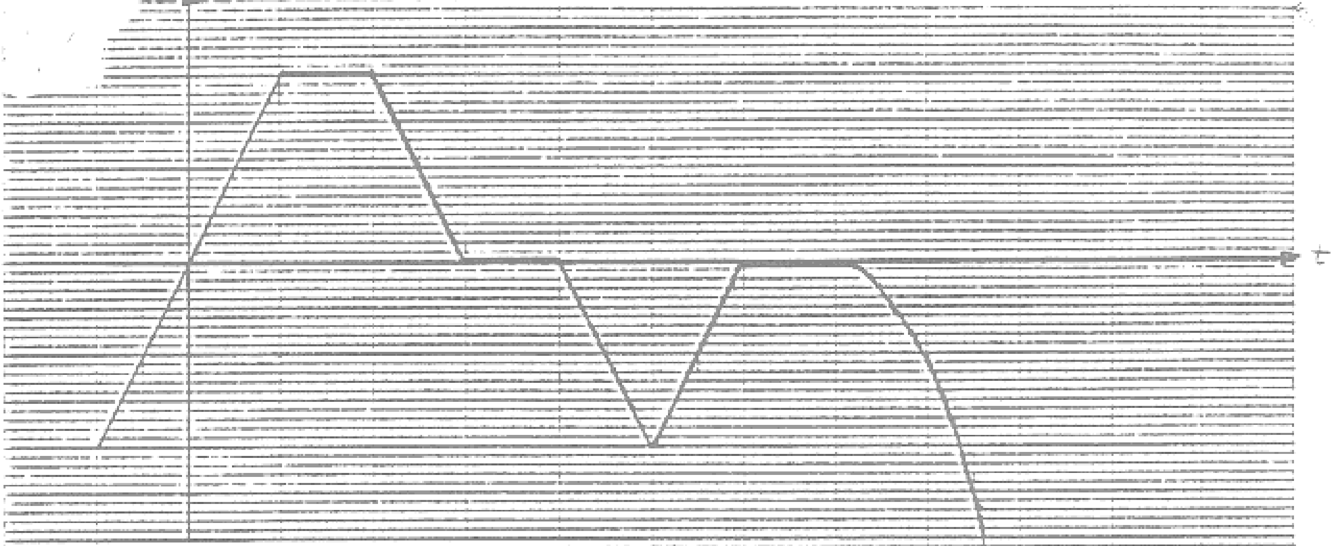
To get practice on visualizing their relationships to each other, please work on the following exercise.

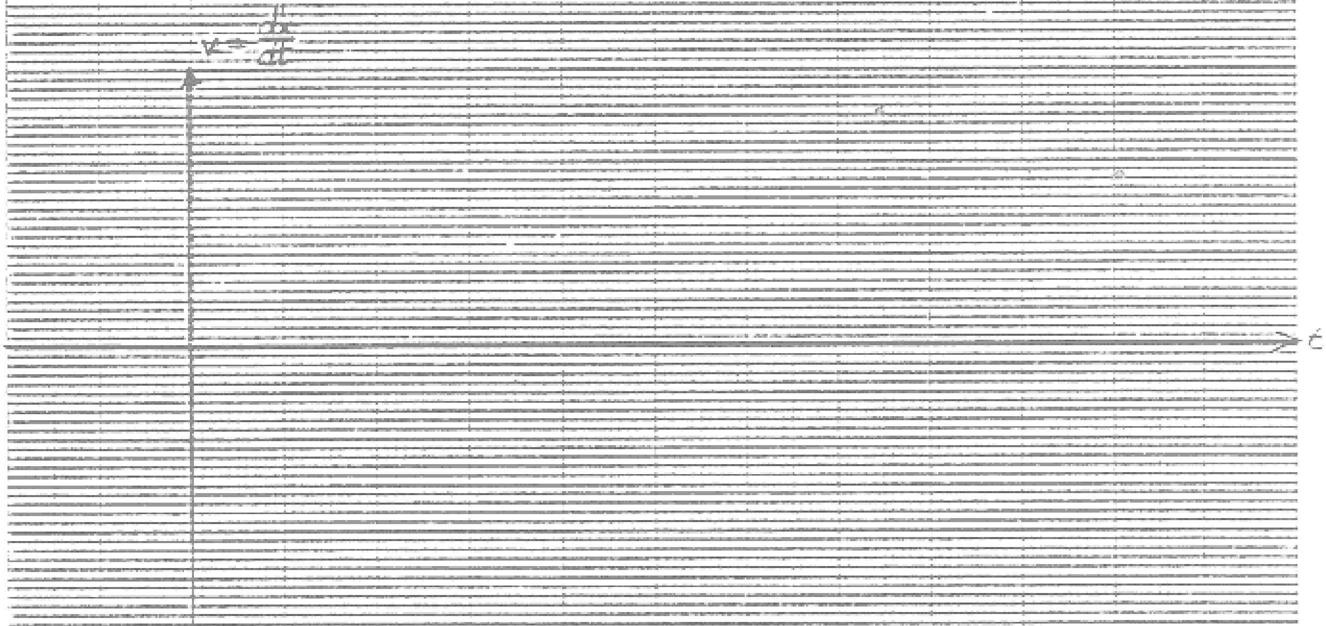


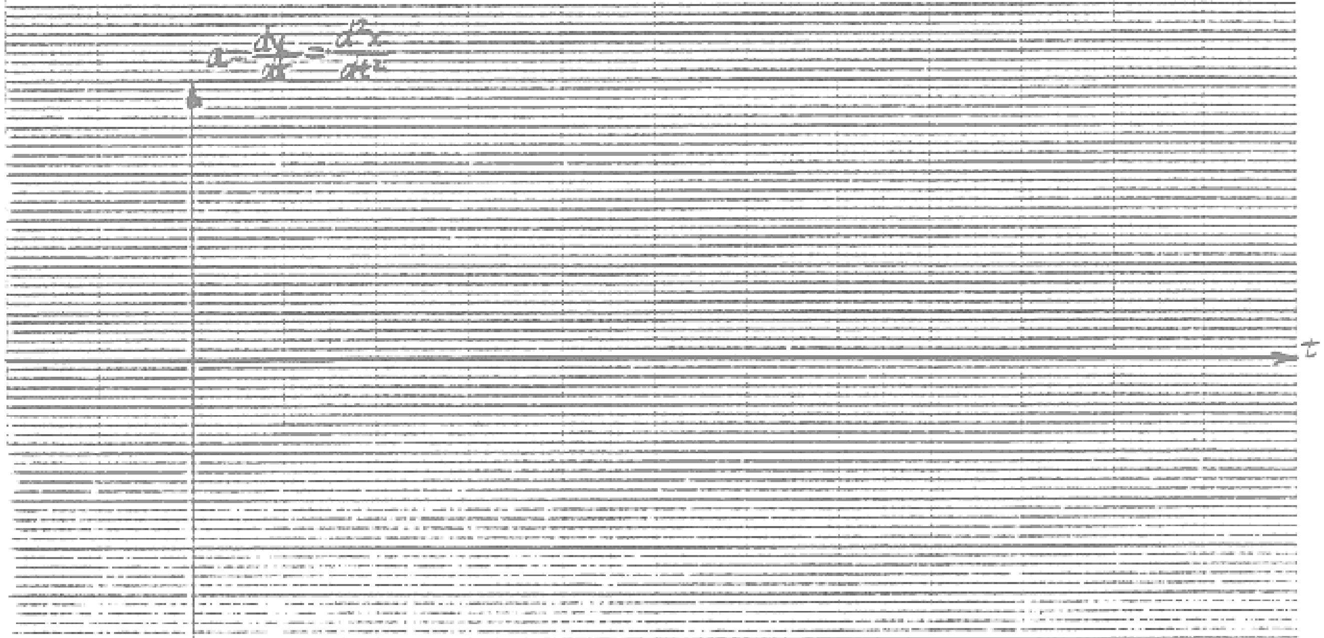
x = -2t
v = dx/dt
v = 2
DO
a = dv/dt = d²x/dt²
a = 0
t

↓ THIS IS A BONUS QUESTION






Squares to the Inch