PHY 209
Space and Time in Elementary Physics

Summary and Applications of Differential Calculus

Review of Differential Calculus

Recall that

$$\frac{d}{dt}(At^n) = Ant^{n-1},$$

where A is a constant.

Some examples:

$$\frac{d}{dt}(\text{constant}) = 0$$

$$\frac{d}{dt}(t) = 1$$

$$\frac{d}{dt}(At^2) = 2At$$

$$\frac{d}{dt}(At^3) = 3At^2$$

which is valid for all possible n, whether a positive integer, zero, negative, or even a non-integer. For example, for $n = \frac{1}{2}$:

$$\frac{d}{dt}(t^{\frac{1}{2}}) = \tfrac{1}{2}t^{\left(\frac{1}{2}-1\right)} = \tfrac{1}{2}t^{-\frac{1}{2}}$$

In other words, using $t^{\frac{1}{2}} = \sqrt{t}$,

$$\frac{d}{dt}(\sqrt{t}) = \frac{1}{2\sqrt{t}}$$

Also, hopefully, you learned from "plotting a graph of the slopes" that $\frac{d}{dt}(\sin t) = \cos t$ and $\frac{d}{dt}(\cos t) = -\sin t$. We could have obtained these expressions from the definition of the derivative. However, I felt that it was more important for you to get a feeling for what the derivative *means* first.

More generally,

$$\frac{d}{dt}(A\sin\omega t) = A\omega\cos\omega t \qquad \frac{d}{dt}(A\cos\omega t) = -A\omega\sin\omega t.$$

To see this where this factor of ω comes from, we could also follow the definition. But here is a quicker way:

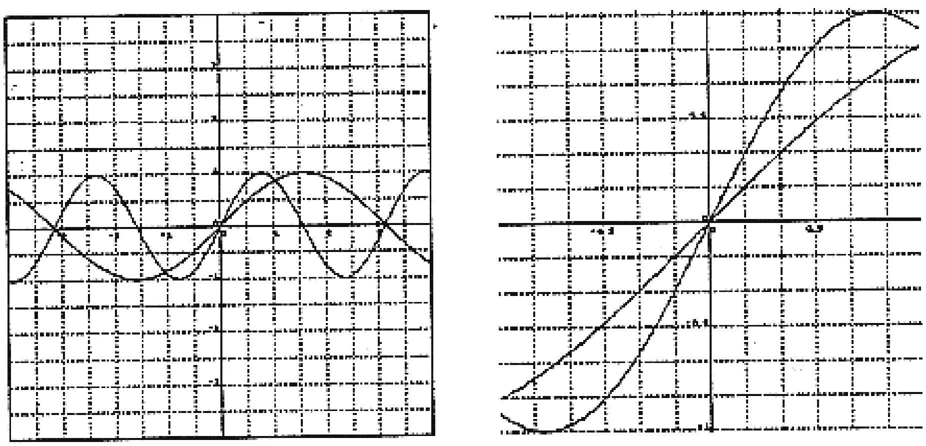
We already know that

$$\frac{d}{du}(A\sin u) = A\cos u,$$

where we use the running-variable u. (Don't let the use of an alternate symbol throw you... they're all just symbols anyway. What *is* important is the "functional relationship" between this function and its derivative.)

Now let us replace the variable u by ωt, where ω is a constant and t is running another variable:

$$\begin{aligned}
\frac{d}{du}(A\sin u) &= A\cos u \\
\frac{d}{d\,[\omega t]}(A\sin[\omega t]) &= A\cos[\omega t] \\
\frac{1}{\omega}\frac{d}{dt}(A\sin\omega t) &= A\cos\omega t \\
\frac{d}{dt}(A\sin\omega t) &= A\omega\cos\omega t
\end{aligned}$$



$y = \sin x$ and $y = \sin 2x$, normal and zoomed-in

An application

One big application of calculus is **Optimization**—in other words, finding the optimum conditions for some situation.

Consider this problem:

Suppose we know that the position of a projectile obeys

$$y = y_0 + v_0 t - \frac{1}{2}gt^2$$

Problem: Find the value of t that makes the height a maximum—in other words, *"How long do I have to wait until the projectile gets to its maximum height?"*

At the time, t_{max}, when the particle reaches its maximum height, y_{max}, its [vertical] velocity (*i.e.*, its rate-of-change of position with time) *at that moment* is zero. In other words,

$$\left.\frac{dy}{dt}\right|_{t=t_{max}} = 0.$$

Solving this equation for t_{max} will tell us how long we have to wait.

First, calculate the velocity (by taking the derivative of the position with respect to time):

$$\frac{dy}{dt} = v_0 - gt$$

Second, set this equal to zero for the choice $t = t_{max}$ (since when we reach time t_{max}, the velocity is zero):

$$0 = v_0 - gt_{max}$$

Now, solve this equation for t_{max}:

$$t_{max} = \frac{v_0}{g}.$$

This is the time that you need to wait before the projectile reaches its highest point.

Problem: What is the maximum height of the projectile?

We know that the position as a function of time is given by

$$y = y_0 + v_0 t - \frac{1}{2} g t^2$$

and we know (from the previous problem) that the projectile reaches its maximum height at time $t = t_{max} = \frac{v_0}{g}$. So, we simply substitute for t in the first equation:

$$y = y_0 + v_0 \left[\frac{v_0}{g}\right] - \frac{1}{2} g \left[\frac{v_0}{g}\right]^2$$

and reduce the algebra

$$y = y_0 + \frac{{v_0}^2}{g} - \frac{1}{2}\frac{{v_0}^2}{g} = y_0 + \frac{{v_0}^2}{2g}$$

The maximum height is $\frac{{v_0}^2}{2g}$ higher than your starting height y_0.

- Using the Range Formula $R = \frac{2{v_0}^2}{g}\sin(2\theta)$, find θ_{max}, the value of θ that makes R as a maximum. *Hint: Calculate $\frac{dr}{d\theta}$. That expression is equal to zero when R is at its maximum, and thus when $\theta = \theta_{max}$.*

- Find the maximum value for R.

A farmer has 20 meters of fence to enclose a rectangular area. What is the largest rectangular area that the farmer can enclose?

Well, for a rectangle with width x and height y, the area is

$$A = xy.$$

But the perimeter $p = 2x + 2y$ is a constant, namely $p = 20$ meters.

Observe that if I choose $x = 0.1$ and $y = 9.9$, the area is 0.99.
If I choose $x = 1$ and $y = 9$, the area is 9.
If I choose $x = 3$ and $y = 7$, the area is 21.
If I choose $x = 8$ and $y = 2$, the area is 16.
If I choose $x = 9$ and $y = 1$, the area is 9.
But what should I choose to get the largest area?

You could almost guess that the answer is a square with side 5, using your intuition and using symmetry. Let's see how calculus would solve this problem.

So,

$$A = xy \qquad (\text{constant})p = 2x + 2y$$

We can eliminate y from the pair of equations by solving $p = 2x + 2y$ for y

$$y = \tfrac{1}{2}(p - 2x).$$

So,

$$A = xy = x(\tfrac{1}{2}(p - 2x)) = \tfrac{1}{2}px - x^2.$$

To get the maximum possible value of A, we take the derivative of this function with respect to x and find the particular [critical] value of x that makes the derivative zero at that value. So,

$$\begin{aligned} \frac{d}{dx}A &= \frac{d}{dx}(\tfrac{1}{2}px - x^2) \\ &= \frac{d}{dx}(\tfrac{1}{2}px) - \frac{d}{dx}(x^2) \\ &= \tfrac{1}{2}p - 2x \end{aligned}$$

Recall that this tells us the slope of the graph of A vs. x. Note that, with $p = 20$, $\frac{dA}{dx} > 0$ for $x < 5$ and $\frac{dA}{dx} < 0$ for $x > 5$. In other words, as we increase x from 0 towards 5, the area is increasing (hence, the positive sign of the derivative); as we increase x from 5 towards 10, the area is now decreasing (hence, the negative sign of the derivative). It must have been zero somewhere.

Setting $\frac{dA}{dx} = 0$,

$$0 = \frac{d}{dx}A = \tfrac{1}{2}p - 2x$$

$$x = \frac{p}{4}$$

Hence, we find that when $x = \frac{p}{4} = 5$, that $\frac{dA}{dx} = 0$ there, and, therefore, that A has reached its maximum value. Thus, we should choose $x = 5$ to get the largest area.

But what is the corresponding value for y? Go back to the equation for the perimeter $p = 2x + 2y$ (or equivalently, $y = \frac{1}{2}(p - 2x)$). With $x = 5$ (and $p = 20$), we find that $y = 5$. Hence, given p, we should form a square of side $\frac{p}{4}$ to enclose the largest area.

Bonus (1 pt)

- A tin cylindrical can (with radius r and height h) is to enclose a fixed volume v. How should we choose r (and h) such that we use the least amount of tin? (*Hint: the volume of a cylindrical can is $v = \pi r^2 h$, and the total surface area of the tin can is $A = 2\pi rh + 2 \cdot \pi r^2$ (the first term is the lateral region, the second term are the two lids of the can).* You might run into a cube-root along the way, but don't worry. If all turns out okay, you should get $r = h/2$.) You must show all your work to get credit.

Something profound in the Physical World

For some reason, which we don't understand, Nature likes to "optimize". In physics, there is a very powerful principle called the **Principle of Least Action.** Nearly every physical system has a physical quantity called the **Action** of that system. The principle of least action states that **a physical system will behave to as to minimize this Action.**

After Newton wrote down his law of motion for particles $F = ma$, other physicists and mathematicians discovered this principle which, suprisingly, has Newton's formula as a special case. In fact, this principle can come disguised as **Fermat's Principle of Least Time** for optics which says that **light will travel between two points in a way which takes the shortest time.** If there is time in class today, I will show you (using basic calculus) how the laws of geometrical optics, namely the **Law of Reflection** and the **Law of Refraction,** fall out of this principle.