PHY 209
Space and Time in Elementary Physics

Energy and the Harmonic Oscillator plus a Glimpse of Integral Calculus

Recall that for a particle undergoing a constant acceleration in a straight line, its position x at any time t is given by

$$\boxed{x = x_0 + v_0 t + \tfrac{1}{2}at^2}$$

Now, by taking the derivative of this equation with respect to time t, its velocity v at any time t is given by

$$v = v_0 + at.$$

We can solve this last equation for t, to obtain a mathematically-equivalent equation

$$\boxed{t = \frac{v - v_0}{a}}$$

Just for fun, let us eliminate t from the first equation by substituting this expression for t:

We did a similar calculation yesterday.

$$\begin{aligned}
x &= x_0 + v_0 t + \tfrac{1}{2}at^2 \\
x - x_0 &= v_0 t + \tfrac{1}{2}at^2 \\
&= v_0\left[\frac{v - v_0}{a}\right] + \tfrac{1}{2}a\left[\frac{v - v_0}{a}\right]^2 \\
&= v_0\left[\frac{v - v_0}{a}\right] + \tfrac{1}{2}\frac{(v - v_0)^2}{a} \\
&= \left(\frac{v - v_0}{a}\right)\left(v_0 + \tfrac{1}{2}(v - v_0)\right) \\
&= \left(\frac{v - v_0}{a}\right)\left(v_0 + \tfrac{1}{2}v - \tfrac{1}{2}v_0\right) \\
&= \left(\frac{v - v_0}{a}\right)\left(\tfrac{1}{2}v_0 + \tfrac{1}{2}v\right) \\
&= \tfrac{1}{2}\frac{(v - v_0)(v + v_0)}{a} \\
&= \tfrac{1}{2}\left(\frac{v^2 - {v_0}^2}{a}\right)
\end{aligned}$$

So,

$$\boxed{a(x - x_0) = \tfrac{1}{2}(v^2 - {v_0}^2)}$$

Multiplying both sides by the mass m of the particle, we obtain

$$ma(x - x_0) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}m{v_0}^2.$$

Recall the Newton Law of Motion: $\vec{F} = m\vec{a}$. From this, the magnitude of the acceleration, a, is

$$\boxed{a = \frac{F}{m}}$$

Since the acceleration a is constant, the force F is also a constant.

Substituting for a,

$$m\left[\frac{F}{m}\right](x - x_0) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}m{v_0}^2$$

which reduces to

$$\boxed{F \cdot (x - x_0) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}m{v_0}^2}$$

The quantity on the left is the product of "the constant applied-force F" and "the displacement $(x - x_0)$ from the starting-position x_0 to the final-position x". It is **a measure of the effect of the applied-force F**. In fact, it has a special name—the <u>work</u> done by the force.

The quantity on the right is very interesting. **Why?** It seems that "$\frac{1}{2}mv^2$" **is a special measure of the state of motion of an object.** In fact, it, too, has a special name—the **<u>kinetic-energy</u>** of the object. Thus, the quantity on the right is "the difference between the final kinetic-energy and the initial kinetic-energy".

* It is very interesting because *both terms of this difference are of the form* **"one-half the mass times the square of the velocity".**

The equation is interesting because:
Qualitatively, it says that

> **"the application of a force on an object has affects the state-of-motion of that object".**

Quantitatively, it says that

> **"the work done by an applied-force is equal to the change in the kinetic-energy of the object".**

The Falling Apple—Motion Under a Constant Force

Let's consider the following situation: An apple of mass m falling from a tree of height h.

Initially at rest ($v_0 = 0$), it falls from a height $x_0 = h$. It hits the ground ($x = 0$) with velocity v. There is one force at work: the force due to gravity. This force has magnitude mg but points downward (so, it picks up a minus sign). The "work done by gravity" would be $-mg \cdot (0 - h)$. The "change in kinetic-energy" would be $\frac{1}{2}mv^2 - \frac{1}{2}m[0]^2$. If we reduce the algebra, we have:

$$\text{"the work done" } mgh = \text{"the change in kinetic-energy" } \tfrac{1}{2}mv^2.$$

What this means is that *the force of the earth's gravitational attraction has done work on the apple, which results in a change in the apple's kinetic-energy.*

Kinetic-Energy and Potential-Energy

Kinetic-energy, usually denoted by K, can be thought of as **"the energy of motion"**. Indeed, $K = \frac{1}{2}mv^2$ makes reference to motion because of its dependence on the velocity v of the object.

In special cases, there is another notion of energy: *potential-energy*.

Such a special case occurs, for example, when we deal with the force of gravity. The force of gravity is a special kind of force—it is an example of what is called a **conservative force**. It turns out that *if a force is conservative*, one can define the notion of **"an energy of (relative) position"** called the **potential-energy**, usually denoted by U. In some sense, it is a stored-up form of energy—it has the *potential* to do work on an object. In the case of gravity, the potential-energy is

$$U_{grav} = mgx,$$

where x is the height above the ground.

Due to the force of the earth's gravity, it takes work to raise an apple from the ground to a height $x = h$. It takes work because I have to oppose the force due to gravity as I pick up the apple and raise it to a height $x = h$. In my applying a force over that distance, I have effectively stored energy in it.

How much? I have effectively stored $U = mgh$ in the apple.

Why? Because, if I release it from rest at that height $x = h$, it will not remain at rest. Instead, it will return to the ground under the influence of the earth's gravity—a constant downward force of magnitude mg will act over a displacement h.

So, one can think of *"gravitational potential-energy"* $U_{grav} = mgx$ as

"the work the force of gravity would apply over the displacement x"

In effect, the potential energy stored-up in it due to its position (height) would be converted into the energy of motion—into kinetic energy. If no energy is lost in this conversion, one says that **the total energy is conserved**.

The Harmonic Oscillator—Motion Under a Linear Force

Let's now consider the following situation: a mass m hanging from a spring (with spring constant k).

Whereas the force due to the earth's gravity is constant (near the earth's surface), the force due to a spring is *not constant*. A mass at rest, hanging from a spring, is at its **equilibrium position**, x_{eq}, which is often set to be zero. In this situation, the force of gravity and the force of the spring balance to yield a zero net-force on the mass. Thus, by Newton's Laws, the mass is unaccelerated, and, if started at rest, it remains at rest.

However, if we displace the mass away from its equilibrium position, so as to stretch or compress the spring, we find that the spring applies a force with **direction opposite to the displacement**, with magnitude **linear in** (*i.e.*, **"proportional to"**) **the displacement from equilibrium**, x. This is known as Hooke's Law. In symbols,

$$F_{spring} = -kx,$$

where k is the proportionality constant, called the **spring-constant**, which depends on the material composing the spring. The minus sign tells us that the force is directed *opposite to the displacement.*

So, using Hooke's force-law in Newton's Law of Motion:

$$[-kx] = ma$$

However, recall that the acceleration $a = \frac{d}{dt}v = \frac{d}{dt}\left(\frac{d}{dt}x\right)$. [Recall that this means that a is the slope-function of the graph of v-vs.-t, where v is the slope-function of the graph of x-vs.-t.] Notice that **the acceleration is not a constant**—it is proportional to the displacement:

$$a = -\frac{k}{m}x$$

In fact, it is opposite to the displacement x, with proportionality constant $\frac{k}{m}$.

While this relation is true, as written, it is not very useful for us if we wish to find the position from equilibrium x as a function of time t. The way to proceed is to express the acceleration-function, a, in terms of the second-derivative of the position-function, x:

$$-kx = m\frac{d}{dt}\left(\frac{d}{dt}x\right).$$

It becomes a mathematical problem to discover its solution (*i.e.*, the form of the position-function $x(t)$ which will satisfy the equation). We could, but won't, proceed along these lines.

Instead, we will determine the solution experimentally.

- What is the form of the position-function $x(t)$?

- What is the velocity-function $v(t)$?

- What is the acceleration-function $a(t)$?
  Is it proportional to the position-function $x(t)$?

The Kinetic-Energy and Potential-Energy for the Harmonic Oscillator

Assume that the mass is given symbolically as m and that the spring-constant is given symbolically as k.

Since you now know the velocity-function $v(t)$, you can easily find the kinetic-energy function for the harmonic oscillator:

$$K_{spring} = \frac{1}{2}mv^2$$

- What is the kinetic-energy of a harmonic oscillator?

Finding the potential-energy of the harmonic oscillator is a little harder. Recall the interesting relation:

$$\boxed{F \cdot (x - x_0) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}m{v_0}^2}$$

We derived it under the assumption of motion under *constant acceleration* (and, therefore, a constant force), which is not the case for the harmonic oscillator.

However, this relation is *almost correct* for the harmonic oscillator.

With F a constant, observe that quantity on the left, $F \cdot (x - x_0)$ is the area of a rectangle with height F and base $(x - x_0)$ on a graph of force F vs. position x.

More generally, the quantity on the left is **the area between the force-function F and the x-axis.**

- Suppose you start from the equilibrium position ($x_0 = 0$) and then stretch the spring to a displacement x. The force you have to apply is $F_{you} = kx$ (since you have to oppose the force due to the spring $F = -kx$). What is the area between this line and the x-axis?

This area represents the amount of potential energy U_{spring} that you store in the spring by stretching it.

You can consider "the area-function" as a function of x, which measures "the area between the force-function F and the x-axis **with base x**". You essentially did this in the last question!

Now, for something interesting...

- Consider a graph of the area-function vs x. Find the slope-function of this curve. Notice anything special?

A Glimpse of Integral Calculus

There's not much time to go into it here... but allow me to tell you something interesting.

Let f be a function of the running-variable x. Recall that **the derivative of that function f with respect to x** can be interpreted as the slope-function of f on an f-vs.x graph.

Here, we learn that if we consider "the area-under-f"-function on an "area-function"-vs.-x graph. The *"area-function"*'s slope-function (not the slope-function of f itself, but the slope-function of the "area-under-f"-function) is again f!

It turns out that the "area-under-f"-function is related to the *integral of f with respect to x*. The integral is another aspect of calculus, called *duh!* **integral calculus**. "Integral" refers to "addition" or "summation". In some sense, the "derivative" and the "integral" are inverse operations.

The derivative of "the integral of f" is f:

$$\boxed{\frac{d}{dx}\left(\int f\, dx\right) = f}$$

As an example, let $f = kx$.
You essential already found that

$$\text{area-function of } f = \int f\ dx = \int [kx]\ dx = \frac{1}{2}kx^2.$$

The derivative of this is

$$\frac{d}{dx}\left(\int f\ dx\right) = \frac{d}{dx}\left[\frac{1}{2}kx^2\right] = \frac{1}{2}2kx = kx = f$$

Another example, let $f = C$ (a constant, like mg).

$$\text{area-function of } f = \int f\ dx = \int [C]\ dx = Cx.$$

The derivative of this is

$$\frac{d}{dx}\left(\int f\ dx\right) = \frac{d}{dx}[Cx] = C = f$$

In fact, a general rule for functions of the form $f = x^n$:

$$\boxed{\text{area-function of } f = \int f\ dx = \int [x^n]\ dx = \frac{1}{n+1}x^{n+1}.}$$

The derivative of this is

$$\frac{d}{dx}\left(\int f\ dx\right) = \frac{d}{dx}\left[\frac{1}{n+1}x^{n+1}\right] = \frac{1}{n+1}\frac{d}{dx}\left[x^{n+1}\right] = \frac{1}{n+1}\left[(n+1)x^{(n+1)-1}\right] = x^n = f$$

In fact, it almost works in the opposite order:

The integral of "the derivative of f" is f (plus a possibly-zero constant C):

$$\boxed{\int\left(\frac{d}{dx}f\right)\ dx = f + C}$$

Let's just check this for $f = x^n$:

$$\int\left(\frac{d}{dx}f\right)\ dx = \int\left(\frac{d}{dx}[x^n]\right)\ dx = \int\left[nx^{n-1}\right]\ dx$$

$$= n\int\left[x^{n-1}\right]\ dx = n\left[\frac{1}{(n-1)+1}\left(x^{(n-1)+1} + C\right)\right] = n\left[\frac{1}{n}(x^n + C)\right] = x^n + C = f + C$$

• **Bonus:** What is $\int \sin x\ dx$ and $\int \cos x\ dx$?