PHY 209
Space and Time in Elementary Physics

Momentum and Energy

Recall that for a particle undergoing a constant acceleration in a straight line, its position x at any time t is given by

$$x = v_0 t + \tfrac{1}{2}at^2,$$

and (by taking the derivative of this equation with respect to time t) its velocity v at any time t is given by

$$v = v_0 + at.$$

Recall the Newton Law of Motion: $\vec{F} = m\vec{a}$. From this, the magnitude of the acceleration, a, is

$$a = \frac{F}{m}.$$

Thus, we may express the velocity as

$$v = v_0 + \frac{F}{m}t.$$

Multiply through by m to get

$$mv = mv_0 + Ft,$$

which we can write as

$$Ft = mv - mv_0.$$

This quantity Ft is an interesting one because it is equal to the change in "the mass times the velocity" between the current time t and the initial time $t = 0$. This "mass times velocity" quantity is called the **momentum** of the particle. This difference, Ft, is called the **impulse** of the force on the particle. MOMENTUM

TIME-OUT: Physical Interlude—Impulse and Momentum

Let's try to understand what these quantities mean physically.

The impulse is like a "nudge that lasts for a time t" that a force gives to a particle. For instance, in the absence of gravity, an apple of mass m initially at rest ($v_0 = 0\text{m/s}$) remains at rest (Newton's Law of Inertia). But if we could switch on gravity for one second, then at that instant the particle would be a distance $4.9\text{m}(= \frac{1}{2}at^2)$ away and be moving with velocity $9.8\text{m/s}(= at)$. If we could then shut off gravity again, then the apple would would continue to move with velocity 9.8m/s. The impulse would be $m[9.8\text{m/s}^2] \cdot [1\text{sec}] = m([9.8\text{m/s}] - [0\text{m/s}])$.

As another example, suppose that our apple of mass m is travelling with velocity v_0 and we want to bring it to rest. To do so, we need to give an impulse of $-mv_0$ (since $Ft = mv - mv_0 = 0 - mv_0 = -mv_0$). The minus sign tells us that we will apply a force $\vec{F}$ in the direction opposite to the velocity, which makes sense because we want to slow it down. From a practical standpoint, we have a choice to make here. I need $Ft = -mv_0$. Observe that I can choose, e.g., "a large F and a small t" or "a small F and a large t". If I care about the apple, then I would probably want to choose a smaller F and let it act for a longer time t. Of course, it would travel a longer distance before it came to rest. (Think of stopping it with a net.) If I don't care about the apple, then I would probably want to choose a larger F and let it act for a shorter time t. It would now travel a shorter distance before it came to rest. (Think of stopping it with a wall.) You can dream up other examples like applying car-brakes, how an air-bag works, whether to land (from a jump) straight-legged vs bent-legged, *etc.*)

Now, let's consider basketball and a cannonball, which we will drop from the Leaning Tower of Pisa. Ignoring wind resistance, we know that the two balls will fall with the same acceleration and will reach the ground at the same time.

- Given a choice, would you rather try to catch the basketball or the cannonball? (Don't write anything. Think about it.)

Hopefully, you'd pick the basketball (assuming, of course, that the basketball is lighter than the cannonball). Why the basketball? It is because it has less momentum than that of the cannonball. That is, even though the v's are the same the mv's are not.

If we had more time, we could discuss more about momentum. But since you'll be in PHY211 soon, you'll have more of an opportunity to discuss this.

- The symbol $\vec{p}$ is the usual symbol for momentum ($\vec{p} = m\vec{v}$). When Newton wrote down "$\vec{F} = m\vec{a}$", he did not write down $force = (mass)(acceleration)$. Instead, he wrote down something more like

$$force = \frac{d}{dt}(momentum).$$

  Show that this reduces to "$F = ma$", assuming m is a constant.

TIME-IN

So, before the interlude, we found

$$Ft = mv - mv_0.$$

Solving for t, we have

$$t = \frac{m}{F}v - \frac{m}{F}v_0 = \frac{m}{F}(v - v_0).$$

Just for fun, let's substitute this expression for t and the expression $a = \frac{F}{m}$ for the acceleration into

$$\begin{aligned}
x &= v_0 t + \tfrac{1}{2}at^2 \\
&= v_0\left[\frac{m}{F}(v - v_0)\right] + \tfrac{1}{2}\left[\frac{F}{m}\right]\left[\frac{m}{F}(v - v_0)\right]^2 \\
&= \frac{m}{F}v_0(v - v_0) + \tfrac{1}{2}\frac{m}{F}(v - v_0)^2 \\
&= \frac{m}{F}(v - v_0)(v_0 + \tfrac{1}{2}(v - v_0) \\
&= \frac{m}{F}(v - v_0)(v_0 + \tfrac{1}{2}v - \tfrac{1}{2}v_0) \\
&= \frac{m}{F}(v - v_0)(\tfrac{1}{2}v + \tfrac{1}{2}v_0) \\
&= \tfrac{1}{2}\frac{m}{F}(v - v_0)(v + v_0) \\
&= \tfrac{1}{2}\frac{m}{F}(v^2 - v_0^2)
\end{aligned}$$

which we can write as

$$Fx = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2.$$

This quantity Fx is also an interesting one because it is equal to the change in "one-half the mass times the squared-velocity" between the current time t and the initial time $t = 0$ (*i.e.*, the current position x and the initial position $x = 0$). This "one-half the mass times the squared-velocity" quantity is called the **kinetic energy** of the particle. This difference, Fx, is called the work done by the force on the particle. KINETIC ENERGY

Let's consider the situation of an apple of mass m falling from a tree of height h. Initially at rest ($v_0 = 0$), it falls a distance h before it hits the ground with velocity v. The "work done by gravity" would be $mg \cdot h = \frac{1}{2}mv^2 - \frac{1}{2}m[0\text{m/s}]^2$). (Technically, work is $\vec{F} \cdot \vec{x}$, and so the left-hand side is really $(-mg\hat{y}) \cdot (-h\hat{y}) = mgh$. Recall that the force of gravity is $-mg\hat{y}$ and that since we *start from up and travel down a distance* h we have $-h\hat{y}$.)

- The symbol T is the usual symbol for kinetic energy ($T = \frac{1}{2}mv^2$). Express the kinetic energy only in terms of the mass m and the momemtum p. (There should be no v's in the expression.)

Tomorrow, we will continue this discussion of momentum and energy. In particular, we will discuss the notion of the "conservation" of momentum and the "conservation" of energy.