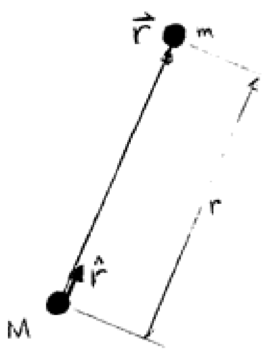
PHY 209
Space and Time in Elementary Physics

Newtonian Gravitation

There is a fable associated with Isaac Newton. Supposedly, the falling of an apple from a tree led Newton to ask if the agent responsible for that apple to fall is the same that responsible for the motion of the moon about the earth. I don't know whether this is a true story or not. But what I do know is that Newton eventually formulated the celebrated Law of Universal Gravitation:



$$\vec{F} = -G\frac{Mm}{r^2}\hat{r}$$

> *The force $\vec{F}$ that a point mass m feels due to another point mass M located at position $\vec{r}$ (from mass M) has magnitude $F = G\frac{Mm}{r^2}$ (where G is a constant of nature) and direction towards the mass m.*

The constant G is called the **Newtonian graviational constant**, and it has a value of $6.672 \times 10^{-11}\mathrm{Nm}^2/\mathrm{kg}^2$. The vector $\hat{r}$ is the unit-vector drawn from mass M to mass m. (Recall that $\vec{r} = r\hat{r}$, "$\vec{r}$ has magnitude r and points in the direction of $\hat{r}$".) The minus sign tells us that the force is in the opposite direction of $\hat{r}$. In physical terms, this says that **gravity is an attractive force.**

rule of thumb: $G \approx \frac{2}{3} \times 10^{-10}\mathrm{Nm}^2/\mathrm{kg}^2$

It is *universal* in the sense that this law applies to all pairs of objects, *e.g.*, the-apple-and-the-earth or the-moon-and-the-earth. One interesting thing to note about this force is that it is *not a contact force between two objects.* Instead, it is an **action-at-a-distance force.**

Let's study this equation a little more.

Recall another one of Newton's Laws of Motion: $\vec{F}_{\text{net}} = m\vec{a}$. Consider a body of mass m falling only under the influence of the earth's gravity. Thus,

$$\vec{F}_{\text{net}} = \vec{F} = -G\frac{Mm}{r^2}\hat{r},$$

where M is the mass of the earth, and r is the distance between the center of the earth and the center of the falling body. Using $\vec{F}_{\text{net}} = m\vec{a}$, we have

$M \approx 6 \times 10^{24}\mathrm{kg}$

$$m\vec{a} = \vec{F} = -G\frac{Mm}{r^2}\hat{r}.$$

Cancelling m from both sides, we have

$$\vec{a} = -G\frac{M}{r^2}\hat{r}.$$

Notice that this acceleration does **not** depend on the mass of the falling body. In other words, **all bodies fall at the same rate under the influence of gravity.** This is good since this agrees with Galileo's experimental observation.

But isn't $a = g = 9.8\text{m/s}^2$, a constant that does not depend on how high up on is from the ground? (r, the distance between the bodies, appears in the equation.) Yes, this is true... **but only near the earth's surface.** To see this, observe that if we are near the surface of the earth then $r = R_{\text{earth}} + h$, where R_{earth} is the radius of the earth and h is the height above the ground. Thus,

$R_{\text{earth}} \approx 6.4 \times 10^6\text{m}$

$$\begin{aligned} a &= G\frac{M}{(R_{\text{earth}}+h)^2} \\ &= G\frac{M}{R^2_{\text{earth}}(1+\frac{h}{R_{\text{earth}}})^2} \\ &= G\frac{M}{R^2_{\text{earth}}}\left(1+\frac{h}{R_{\text{earth}}}\right)^{-2} \end{aligned}$$

Now, since we are interested only in points *near the earth's surface*, we are only interested in small h's, where "small" means $h << R_{\text{earth}}$. Thus, we could look at a graph and zoom-in near $r = R_{\text{earth}}$ and see a straight line with slope $-2\frac{h}{R_{\text{earth}}}$. In other words, we have

$$\begin{aligned} a &\approx G\frac{M}{R^2_{\text{earth}}}\left(1-2\frac{h}{R_{\text{earth}}}\right) \\ &\approx G\frac{M}{R^2_{\text{earth}}} - 2G\frac{M}{R^2_{\text{earth}}}\left(\frac{h}{R_{\text{earth}}}\right) \end{aligned}$$

Note that the first term is a constant, independent of h, and that the second term depends on h. However, with $h << R_{\text{earth}}$, this second term is tiny compared to the first term. Right?

- Calculate each term in this last formula for a. Use $h = 10^3\text{m}$ for the second term.

Bonus (2 pts) (We will do this in class tomorrow. But if you can do it now, you can earn some bonus points.)

- An apple falls according to

$$s_{\text{apple}} = \tfrac{1}{2}a_{\text{apple}}t^2.$$

In one second, the apple falls $s_{\text{apple}} = 4.9\text{m}$, using the fact that $a_{\text{apple}} = g = 9.8\text{m/s}^2$ near the earth.

Now, the moon also falls according to

$$s_{\text{moon}} = \tfrac{1}{2}a_{\text{moon}}t^2.$$

Using the information on this handout and the fact that the distance to the moon is about 60 times the radius of the earth, **calculate** how far the moon falls in one second.