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MTH 207 - CALCULUS I EXAM I

Instructions: *Please include all relevant work to get full credit. Write your solutions using proper notations. Encircle your final answers.*

1. Evaluate the limit, if it exists. If the limit does not exist, write ∞, $-\infty$, or **DNE**.

a. $\lim_{x\to -3} \dfrac{2x^2-18}{x^2+2x-3} = \lim_{x\to -3} \dfrac{2(x-3)\cancel{(x+3)}}{\cancel{(x+3)}(x-1)} = \dfrac{2(-6)}{-3-1} = \dfrac{12}{4} = 3$ [6]

b. $\lim_{x\to \infty} \dfrac{2x^2-18}{x^2+2x-3} = \lim_{x\to\infty} \dfrac{2-\frac{18}{x^2}}{1+\frac{2}{x}-\frac{3}{x^2}} = \dfrac{2}{1} = 2$ [6]

c. $\lim_{x\to 2} \dfrac{x-\sqrt{x+2}}{x-2} \cdot \dfrac{x+\sqrt{x+2}}{x+\sqrt{x+2}}$ [8]

$= \lim_{x\to 2} \dfrac{x^2-(x+2)}{(x-2)(x+\sqrt{x+2})} = \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x+\sqrt{x+2})}$

$= \lim_{x\to 2} \dfrac{x+1}{x+\sqrt{x+2}} = \dfrac{3}{2+2} = \dfrac{3}{4}$

d. $\lim_{t\to\infty} \sqrt{t^2+10t}-t \cdot \dfrac{\sqrt{t^2+10t}+t}{\sqrt{t^2+10t}+t}$ [8]

$= \lim_{t\to\infty} \dfrac{(\cancel{t^2}+10t)-\cancel{t^2}}{\sqrt{t^2+10t}+t} = \lim_{t\to\infty} \dfrac{10}{\frac{\sqrt{t^2+10t}}{t}+1}$

$= \lim_{t\to\infty} \dfrac{10}{\sqrt{1+\frac{10}{t}}+1} = \dfrac{10}{1+1} = 5$

e. $\lim_{x\to -1^-} \dfrac{|x-1|}{[\![x]\!]-1} = \dfrac{|-1-1|}{-2-1} = \dfrac{2}{-3}$ [5]

Note : $[\![-1.1]\!] = -2$

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2. Using the $\epsilon - \delta$ definition of the limit, prove that $\lim_{x \to 2} 10 - 3x = 4$. [10]

We want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$|f(x) - L| < \varepsilon$ when $0 < |x - a| < \delta$. That is,

$|(10-3x)-4| < \varepsilon$ ~~⇒~~ when $|x-2| < \delta$.

If $|(10-3x)-4| < \varepsilon \Rightarrow |6-3x| < \varepsilon$

$\Rightarrow |-3(x-2)| < \varepsilon \Rightarrow |-3||x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{3}$

$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{3}$.

Proof: Given $\varepsilon > 0$, we choose $\delta = \frac{\varepsilon}{3}$.

When $|x-2| < \delta \Rightarrow |x-2| < \frac{\varepsilon}{3} \Rightarrow 3|x-2| < \varepsilon$

$\Rightarrow |3x-6| < \varepsilon \Rightarrow |6-3x| < \varepsilon \Rightarrow |(10-3x)-4| < \varepsilon$

$\Rightarrow |f(x) - L| < \varepsilon$. ∎

3. Prove the **Squeeze Theorem**. That is, if $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$. [10]

Proof: The $\lim_{x \to a} f(x) = L$ implies that $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t.

$|f(x) - L| < \varepsilon$ when $|x-a| < \delta_1$. Hence, when

$|x-a| < \delta_1$, $L - \varepsilon < f(x) < L + \varepsilon$.

Also, $\lim_{x \to a} h(x) = L$ implies that $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ s.t.

when $|x-a| < \delta_2$, $L - \varepsilon < h(x) < L + \varepsilon$.

For a given $\varepsilon > 0$, we choose $\delta = \min(\delta_1, \delta_2)$.

$\Rightarrow$ when $|x-a| < \delta$, both $f(x)$ and $h(x)$ will be between

$L - \varepsilon$ and $L + \varepsilon$. Since $f(x) \le g(x) \le h(x)$, then

$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon \Rightarrow L - \varepsilon < g(x) < L + \varepsilon$

$\Rightarrow |g(x) - L| < \varepsilon$. Therefore, $\lim_{x \to a} g(x) = L$.

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4. Use the Intermediate-Value Theorem to show that the equation $x^3 + x = 4x^2 - 3$ has at least one solution between 1 and 2. [8]

Consider the function, $f(x) = x^3 + x - (4x^2 - 3)$.
This function is continuous in $(1,2)$ because it is a polynomial (polynomials are continuous everywhere, $(-\infty, \infty)$).
Also, note that $f(1) = 1+1-(4-3) = 1$ and $f(2) = 8+2-(16-3) = -3$.
Then by the Intermediate Value Theorem, there is a number c between 1 and 2, such that $f(c) = 0$, since 0 is between 1 and -3. This will make $x = c$ a solution of the original equation because
$f(c) = c^3 + c - (4c^2 - 3) = 0 \Rightarrow c^3 + \cancel{x}c = 4c^2 - 3$.

5. Let

$$f(x) = \begin{cases} ax^2 + b & , x \leq 1 \\ 1/x & , x > 1 \end{cases}$$

Find the values of a and b so that $f(x)$ is differentiable at $x = 1$. [10]

For $f(x)$ to be differentiable at $x=1$, $f(x)$ has to be continuous at $x=1$ and $f'_-(x) = f'_+(x)$.

$$\Rightarrow f'_-(x) = 2ax \rightarrow f'_-(1) = 2a$$
$$f'_+(x) = -x^{-2} \rightarrow f'_+(1) = -1$$
$$\Rightarrow 2a = -1 \quad \boxed{a = -\tfrac{1}{2}}$$

$$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\Rightarrow \lim_{x \to 1^-} ax^2 + b = \lim_{x \to 1^+} \frac{1}{x}$$

$$\Rightarrow a + b = 1 \Longrightarrow -\tfrac{1}{2} + b = 1 \Rightarrow \boxed{b = \tfrac{3}{2}}$$

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6. Determine the derivative of the following functions (*You don't have to simplify*):

a. $y = 2\sqrt[3]{2-x}(2x+1)^4$ [7]

$$\Rightarrow \frac{dy}{dx} = 2\sqrt[3]{2-x}\,(4)(2x+1)^3(2) + \frac{2}{3}(2-x)^{-2/3}(-1)(2x+1)^4$$

b. $y = \dfrac{3x+2}{(3-2x)^3}$ [7]

$$\Rightarrow \frac{dy}{dx} = \frac{(3-2x)^3(3) - (3x+2)(3)(3-2x)^2(-2)}{(3-2x)^6}$$

c. $f(t) = e^{(3t^2)}4^{3t}$ [8]

$$\Rightarrow f'(t) = e^{(3t^2)}4^{3t}(3)\ln 4 + e^{(3t^2)}(6t)\cdot 4^{3t}$$

d. $f(t) = \ln(3t^2 \log_2 t)$ $= \ln(3t^2) + \ln(\log_2 t)$ [8]

$$\Rightarrow f'(t) = \frac{6t}{3t^2} + \frac{1}{\log_2 t}\cdot\frac{1}{t}\cdot\frac{1}{\ln 2}$$

$$= \frac{2}{t} + \frac{1}{\log_2 t}\cdot\frac{1}{t}\cdot\frac{1}{\ln 2}$$

7. Find $\dfrac{dy}{dx}$ if $\tan^2(x+y) = \sin^{-1}(x) + \sec^{-1} y$. [10]

$$\Rightarrow 2\tan(x+y)\cdot\sec^2(x+y)\left(1+\frac{dy}{dx}\right) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{y\sqrt{y^2-1}}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[2\tan(x+y)\sec^2(x+y) - \frac{1}{y\sqrt{y^2-1}}\right] = \frac{1}{\sqrt{1-x^2}} - 2\tan(x+y)\sec^2(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{1}{\sqrt{1-x^2}} - 2\tan(x+y)\sec^2(x+y)}{2\tan(x+y)\sec^2(x+y) - \frac{1}{y\sqrt{y^2-1}}}$$

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8. If $x^2y^2 = 2$, $y = f(x)$, show that $\dfrac{d^2y}{dx^2} = y^3$. [10]

$$\Rightarrow x^2\left(2y\frac{dy}{dx}\right) + 2xy^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xy^2}{2x^2y} = -\frac{y}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x\left(-\frac{dy}{dx}\right) - (-y)(1)}{x^2} = \frac{x\left(\frac{y}{x}\right)+y}{x^2} = \frac{2y}{x^2}$$

From the initial statement, $x^2y^2 = 2 \Rightarrow x^2 = 2/y^2$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2y}{x^2} = \frac{2y}{(2/y^2)} = y^3 \quad \blacksquare$$

9. If an object is thrown vertically, its height (in feet) after t seconds is $H(t) = -16t^2 + v_0t + h_0$, where v_0 is the initial velocity of the object and h_0 is its initial height from the ground. Suppose a ball in thrown vertically upward with an initial velocity of $v_0 = 32$ ft/sec from the top of a building that is 128 ft high. $\Rightarrow H(t) = -16t^2 + 32t + 128$

a. What is the velocity of the ball 2 seconds later? [6]

$$\Rightarrow H'(t) = -32t + 32$$

$$\Rightarrow H'(2) = -32(2) + 32 = -32 \text{ ft/sec}.$$

b. When will the ball hit the ground? What is the velocity of the ball when it hits the ground? [8]

The ball will hit the ground when $H(t) = 0$

$$\Rightarrow -16t^2 + 32t + 128 = 0$$

$$\Rightarrow -16(t^2 - 2t - 8) = 0$$

$$(t-4)(t+2) = 0$$

$$t = 4 \mid t = -2$$

Hence the ball will hit the ground after 4 seconds.

$$\Rightarrow V(4) = H'(4) = -32(4) + 32 = -96 \text{ ft/sec}.$$

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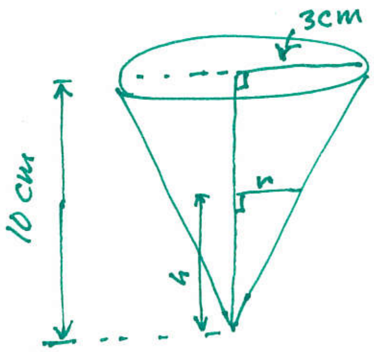
10. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/sec, how fast is the water level rising when the water is 5 cm deep? [Note: Volume of a cone is $V = \frac{1}{3}\pi r^2 h$] [10]



By properties of similar triangles,

$\Rightarrow \frac{r}{h} = \frac{3}{10} \Rightarrow r = \frac{3}{10} h$

$\Rightarrow V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{3}{10} h\right)^2 h$

$= \frac{3}{100} \pi h^3$

Given:

$\frac{dv}{dt} = 2 \text{ cm}^3/\text{sec}$ $\Rightarrow \frac{dv}{dt} = \frac{3}{100} \pi \, 3h^2 \cdot \frac{dh}{dt}$

Required:

$\frac{dh}{dt}$ when $h = 5$ cm $\Rightarrow 2 \frac{\text{cm}^3}{\text{sec}} = \frac{3}{100}\pi (3)(5^2) \cdot \frac{dh}{dt}$ cm²

$\Rightarrow \frac{dh}{dt} = \frac{2 \text{ cm}^3/\text{sec}}{\frac{9}{4}\pi \text{ cm}^2} = \frac{8}{9\pi}$ cm/sec.

$\approx .28$ cm/sec.

11. Find the linearization $L(x)$ of $f(x) = \sqrt[3]{1 + 6x}$ at $a = 0$. Then use it to estimate the value of $\sqrt[3]{1.005}$ [10]

$f'(x) = \frac{1}{3}(1+6x)^{-2/3}(6)$

$\Rightarrow f'(0) = \frac{1}{3}(1)^{-2/3}(6) = 2$

$\Rightarrow L(x) = f(a) + f'(a)(x-a)$

$= f(0) + f'(0)x$

$= 1 + 2x$

Note that $\sqrt[3]{1.005} = \sqrt[3]{1 + 0.005}$ $\Rightarrow 6x = .005 \Rightarrow x = \frac{.005}{6}$

$= f(.005/6)$

$\approx L\left(\frac{.005}{6}\right) = 1 + 2\left(\frac{.005}{6}\right) = 1 + \frac{.005}{3}$

$\approx 1 + .001667$

$= 1.001667$

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