max of 155/150

NAME Key

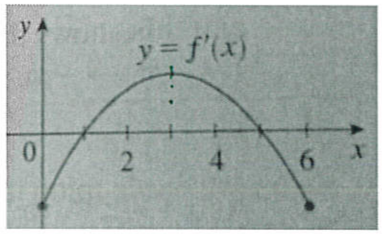
AUGUST 6, 2018

MTH 207 - CALCULUS I

EXAM II

Instructions: *Include all relevant work to get full credit. Write your solutions using proper notations. Encircle your final answers.*

1. Use the graph of the **derivative** of $f(x)$ (shown below) to do following:



a. Construct the first derivative chart that shows the first-order critical values and where $f'(x)$ is positive and negative. Also, include in this chart where the $f(x)$ is increasing and decreasing. Then specify if there ~~is~~ are any local maximum and/or local minimum points. [8]

$f'(x)$ − 0 + 0 −

$f(x)$ dec 1 inc 5 dec

↘ ↗ ↘

⇒ f(x) has a local min at $x = 1$ and local max $x = 5$.

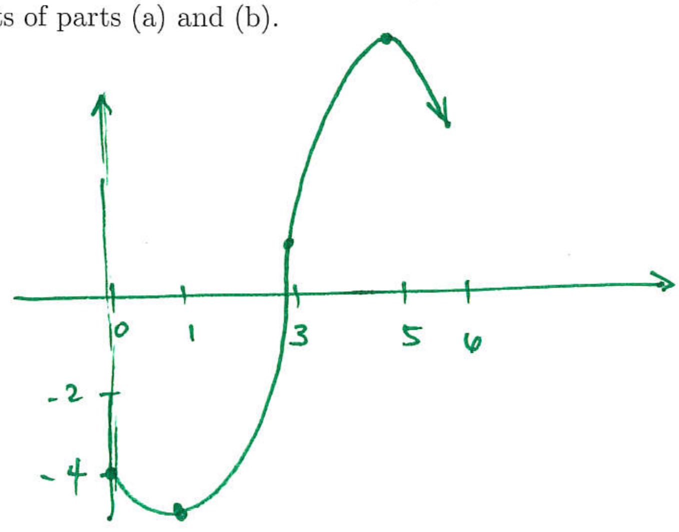
b. Construct the second derivative chart that shows the second-order critical value and where $f''(x)$ is positive and negative. Also, include in this chart where the $f(x)$ is concave up and concave down. Then specify if there is any inflection points. [5]

$f''(x)$ + 0 −

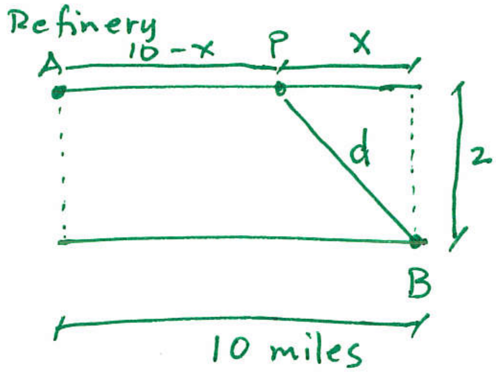
$f(x)$ CU 3 CD

f(x) has an inflection point at $x = 3$.

c. If $f(0) = -4, f(1) = -5, f(3) = 1, f(5) = 7$, sketch the graph of $f(x)$ showing these points and the results of parts (a) and (b). [5]



2. An oil refinery is located on the north bank of a straight river that is 2 miles wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 miles east of the refinery. The cost of laying pipe is \$ 0.5 million dollars per mile over land to a point P on the north bank and \$1 million dollars per mile under the river to the tanks. To minimize the total cost of the pipeline, where should P be located? Check to see that you get a minimum value at this point by evaluating the total cost at the critical value and at the endpoints. [15]



$\Rightarrow d^2 = x^2 + 2^2 \Longrightarrow d = \sqrt{x^2+4}$

Cost: $c(x) = \frac{1}{2}(10-x) + \sqrt{x^2+4}, \quad 0 \le x \le 10$

$\Rightarrow c'(x) = -\frac{1}{2} + \frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{x^2+4}} = 0$

$\Rightarrow \frac{x}{\sqrt{x^2+4}} = \frac{1}{2}$

$\Rightarrow 2x = \sqrt{x^2+4}$

$\Rightarrow 4x^2 = x^2 + 4$

$3x^2 = 4$

$x = \pm\sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

≈ 1.155

$\rightarrow 10 - x = 8.845$

Check:

$c(0) = \frac{1}{2}(10-0) + \sqrt{0^2+4} = 7$

$c(10) = \frac{1}{2}(10-10) + \sqrt{10^2+4}$

$= 0 + \sqrt{104} \approx 10.198$

$c\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{2}\left(10 - \frac{2}{\sqrt{3}}\right) + \sqrt{\frac{4}{3}+4}$

$\approx 6.732 \leftarrow$ abs. min.

Therefore, P should be 8.845 miles east of the refinery.

3. Evaluate $\lim_{x\to\infty} x^{(1/x)}$. [10]

$= \lim_{x\to\infty} e^{\ln x^{1/x}} = e^{\lim_{x\to\infty} \frac{1}{x}\ln x} = e^0 = 1.$

Note: $\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$

4. Given $g(x) = \int_{2x^3}^{5} \sqrt{3t^2+t}\,dt$, find $g'(x)$. [7]

$g'(x) = D_x\left[-\int_5^{2x^3} \sqrt{3t^2+t}\,dt\right] = -\sqrt{3(2x^3)^2 + (2x^3)} * 6x^2$

$= -6x^2\sqrt{12x^6+2x^3}$

5. Evaluate the following integrals:

a. $\displaystyle\int 2\cos x + e^{2x} + \frac{1}{\sqrt{1-x^2}}\,dx$ [8]

$= 2\sin x + \frac{1}{2}e^{2x} + \sin^{-1}(x) + C$

b. $\displaystyle\int_0^1 t(1-t)^2\,dt$ $\displaystyle = \int_0^1 t\,(1-2t+t^2)\,dt$ [8]

$\displaystyle = \int_0^1 (t - 2t^2 + t^3)\,dt$

$\displaystyle = \left(\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\right)\Big|_0^1$

$\displaystyle = \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - (0) = \frac{6-8+3}{12} = \frac{1}{12}$

c. $\displaystyle\int 6x^5\sqrt{1-x^3}\,dx$ $\displaystyle = \int -2x^3\sqrt{1-x^3}\,(-3x^2)\,dx$ [8]

Let $u = 1-x^3$

$du = -3x^2dx$

$\to x^3 = 1-u$

$\displaystyle = \int -2(1-u)\sqrt{u}\,du$

$\displaystyle = \int(-2u^{1/2} + 2u^{3/2})\,du$

$\displaystyle = -2\frac{u^{3/2}}{3/2} + 2\frac{u^{5/2}}{5/2} + C$

$\displaystyle = -\frac{4}{3}(1-x^3)^{3/2} + \frac{4}{5}(1-x^3)^{5/2} + C$

d. $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan\theta}\,d\theta$ [8]

Let $u = 1+\tan\theta$ $\Rightarrow$ when $\theta = 0 \Rightarrow u = 1+\tan(0) = 1+0 = 1$

$du = \sec^2\theta\,d\theta$ when $\theta = \frac{\pi}{4} \Rightarrow u = 1+\tan\left(\frac{\pi}{4}\right) = 1+1 = 2$

$\displaystyle = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

e. $\displaystyle\int_0^{\pi/2} \cos\theta\sin(\sin\theta)\,d\theta$ $\displaystyle = \int_0^1 \sin(u)\,du = -\cos u\Big|_0^1$ [8]

Let $u = \sin\theta$ $\to$ when $\theta = 0 \Rightarrow u = 0$

$du = \cos\theta\,d\theta$ $\theta = \frac{\pi}{2} \Rightarrow u = 1$

$= -\cos(1) + \underbrace{\cos(0)}_{=1}$

$= 1 - \cos(1).$

6. Find $f(x)$ if $f'(x) = \dfrac{\sqrt{x} - x}{x^2}$ and $f(1) = 3$. [10]

$$f(x) = \int \frac{\sqrt{x} - x}{x^2}\,dx = \int (x^{-3/2} - x^{-1})\,dx = \frac{x^{-1/2}}{-1/2} - \ln|x| + C$$

Since $f(1) = 3$, then $\dfrac{(1)^{-1/2}}{-1/2} - \ln 1 + C = 3$

$$\Rightarrow -2 - 0 + C = 3 \Rightarrow C = 5$$

$$\Rightarrow f(x) = -\frac{2}{\sqrt{x}} - \ln|x| + 5$$

7. A ball is thrown upward with an initial velocity of 48 ft/sec from the edge of a cliff 160 ft above the ground. If the only force acting on this ball is Earth's gravity ($a = -32$ ft/sec^2), answer the following questions:

a. Determine the velocity of the ball, $v(t)$, t seconds later. What is the velocity of the ball 1 second later? When will the ball attain it's maximum height? [10]

$$\Rightarrow \frac{dv}{dt} = -32 \Rightarrow v(t) = \int -32\,dt = -32t + C_1$$

when $t=0$, $v(t) = 48 \Rightarrow v(0) = -32(0) + C_1 = 48 \Rightarrow C_1 = 48$

$$\Rightarrow v(t) = -32t + 48.$$

when $t = 1 \Rightarrow v(1) = -32 + 48 = 16$ ft/sec.

The ball will attain its maximum height when $v(t) = 0$

$\Rightarrow -32t + 48 = 0 \Rightarrow t = \dfrac{48}{32} = 1.5$ seconds later.

b. Determine the height function, $h(t)$, t seconds later. When will the ball hit the ground and at what velocity will it hit the ground? [10]

$$\frac{dh}{dt} = v(t) = -32t + 48 \Rightarrow h(t) = \int(-32t + 48)\,dt = -\frac{32t^2}{2} + 48t + C_2$$

when $t=0$, $h(0) = 160 \Rightarrow C_2 = 160$

$$\Rightarrow h(t) = -16t^2 + 48t + 160$$

The ball will hit the ground when $h(t) = 0$

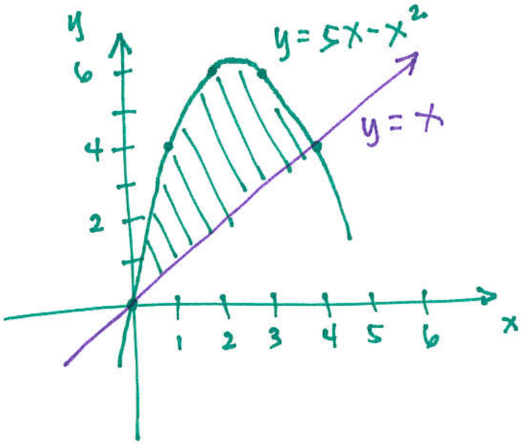
$\Rightarrow -16t^2 + 48t + 160 = 0 \Rightarrow -16(t^2 - 3t - 10) = 0$

$\Rightarrow -16(t-5)(t+2) = 0$

Hence, the ball will hit the ground $t = 5$ | $t = -2$ ← Not possible

5 seconds later with a velocity of $v(5) = -32(5) + 48 = -112$ ft/sec

8. Sketch the region enclosed by the curves $y = x$ and $y = 5x - x^2$, and then calculate the area of the region. [15]



$$5x - x^2 = x \Rightarrow x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x = 0 \mid x = 4$$

$$\text{Area} = \int_0^4 \left[(5x - x^2) - x\right] dx$$
$$= \int_0^4 (4x - x^2)\, dx$$
$$= \left(2x^2 - \tfrac{1}{3}x^3\right)\Big|_0^4$$
$$= \left(32 - \frac{64}{3}\right) - 0$$
$$= \frac{32}{3}$$

9. Prove the **Mean Value Theorem**. That is, if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. [Hint: Use the **Rolle's Theorem**.] [10]

Proof:

Consider the function
$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Since $f(x)$ and $(x-a)$ are both continuous on $[a,b]$, then $h(x)$ is also continuous on $[a,b]$. Also, both $f(x)$ and $(x-a)$ are differentiable on (a,b), hence, $h(x)$ is also differentiable on (a,b). Taking the derivative, we get
$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Finally, note that $h(a) = f(a) - f(a) - \dfrac{f(b) - f(a)}{b - a}(a - a) = 0$ and
$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{\cancel{(b-a)}}\cancel{(b-a)} = 0.$$

Therefore, by Rolle's Theorem, there is a c in (a,b) s.t.
$$h'(c) = 0 \Rightarrow f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

10. Use the **Fundamental Theorem of Calculus - Part I** to prove the **Fundamental Theorem of Calculus - Part II**. That is, if $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$. [10]

Proof:

Let $g(x) = \int_a^x f(t)\,dt$. Then from FTC-I, $g'(x) = f(x)$. Meaning, $g(x)$ is an antiderivative of $f(x)$. Since $F(x)$ is also an antiderivative of $f(x)$, this implies that

$$F'(x) = f(x) = g'(x) \quad \text{on } [a, b].$$

$$\Rightarrow F(x) = g(x) + C, \quad c = \text{constant}.$$

$$\Rightarrow F(b) - F(a) = (g(b) + c) - (g(a) + c)$$

$$= g(b) - g(a)$$

$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0}$$

$$= \int_a^b f(t)\,dt. \quad \blacksquare$$