

**Instructions:** Include all relevant work to get full credit. Write your solutions using proper notations. Encircle your final answers.

1. Evaluate the limit, if it exists. If the limit does not exist, write  $\infty$ ,  $-\infty$ , or DNE.

$$\text{a. } \lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{4x - 2x^2} = \lim_{x \rightarrow 2} \frac{(x-2)(x-1)}{2x(2-x)} \quad [8]$$

$$= \lim_{x \rightarrow 2} \frac{\cancel{(x-2)}(x-1)}{-2x \cancel{(x-2)}} = \frac{1}{-2(2)} = -\frac{1}{4}$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} = \lim_{x \rightarrow 0} \frac{(x+4) - 4}{x(\sqrt{x+4} + 2)} \quad [8]$$

$$= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+4} + 2)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

$$\text{c. } \lim_{x \rightarrow \infty} \sqrt{x^2 + 4x} - x \quad [10]$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 4x} - x}{\sqrt{x^2 + 4x} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 4x) - x^2}{\sqrt{x^2 + 4x} + x} = \lim_{x \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{4}{x}} + 1} = \frac{4}{1+1} = 2$$

2. Let  $h(x) = f(x) + g(x)$ . Prove that  $h'(x) = f'(x) + g'(x)$  using the limit definition of the derivative, assuming both  $f'(x)$  and  $g'(x)$  exist. [10]

Proof:  $h'(x) = \lim_{\Delta x \rightarrow 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) + g(x + \Delta x) - (f(x) + g(x))}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right]$$

$$= f'(x) + g'(x) \quad \square$$

3. Determine the derivative of the following functions (You don't have to simplify):

a.  $y = (2x+1)^3 e^{(3x^2)}$

[8]

$$\frac{dy}{dx} = 3(2x+1)^2(2) e^{3x^2} + (2x+1)^3 e^{3x^2} (6x)$$

b.  $f(\theta) = \left( \frac{e^{4\theta}}{\sin^2(3\theta)} \right)^5$

[8]

$$f'(\theta) = 5 \left( \frac{e^{4\theta}}{\sin^2(3\theta)} \right)^4 \cdot \left[ \frac{\sin^2(3\theta) e^{4\theta} (4) - e^{4\theta} 2 \sin(3\theta) \cos(3\theta) 3}{\sin^4(3\theta)} \right]$$

c.  $f(x) = \ln \left( \frac{3x^2(2-x)^3}{(x^2-1)^2} \right) = \ln(3) + 2\ln(x) + 3\ln(2-x) - 2\ln(x^2-1)$

[10]

$$\Rightarrow f'(x) = 0 + \frac{2}{x} - \frac{3}{2-x} - \frac{4x}{x^2-1}$$

4. If  $x^2 y^2 = 2$ ,  $y = f(x)$ , show that  $\frac{d^2 y}{dx^2} = y^3$ .

[12]

Proof:

$$D_x (x^2 y^2) = 0$$

$$x^2 \left( 2y \cdot \frac{dy}{dx} \right) + 2x (y^2) = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2xy^2}{2x^2 y} = -\frac{y}{x}$$

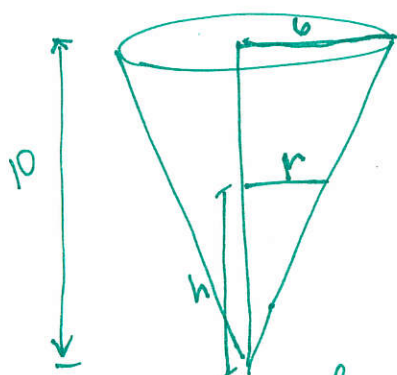
$$\Rightarrow \frac{d^2 y}{dx^2} = - \left( \frac{x \cdot y' - y}{x^2} \right) = \frac{y - x \left( -\frac{y}{x} \right)}{x^2} = \frac{2y}{x^2}$$

$$\text{Since } x^2 y^2 = 2 \Rightarrow x^2 = 2/y^2$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{2y}{(2/y^2)} = y^3 \quad \square$$



6. A paper cup has the shape of a cone with height 10 cm and radius 6 cm (at the top). If water is poured into the cup at a rate of  $2 \text{ cm}^3/\text{sec}$ , how fast is the water level rising when the water is 5 cm deep? [Note: Volume of a cone is  $V = \frac{1}{3}\pi r^2 h$ ] [12]



$$\Rightarrow \frac{r}{h} = \frac{6}{10} \Rightarrow r = \frac{3}{5}h$$

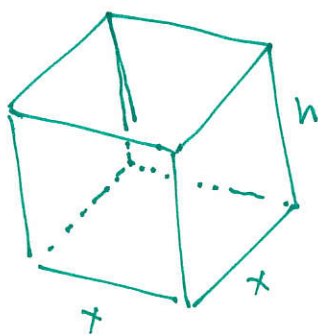
$$\Rightarrow V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{3}{5}h\right)^2 h = \frac{3}{25}\pi h^3$$

$$\frac{dV}{dt} = \frac{9}{25}\pi h^2 \cdot \frac{dh}{dt}$$

when  $h = 5 \text{ cm} \Rightarrow 2 \frac{\text{cm}^3}{\text{sec}} = \frac{9}{25}\pi (5 \text{ cm})^2 \left(\frac{dh}{dt}\right)$

$$\Rightarrow \boxed{\frac{dh}{dt} = \frac{2}{9\pi} \text{ cm/sec}}$$

7. If  $1200 \text{ cm}^2$  of material is available to make a box with a square base and an open top, find the dimensions of the box with the largest possible volume. What is the largest possible volume? Make sure that you check/verify that you get a maximum volume with these dimensions. [15]



$$\Rightarrow x^2 + 4xh = 1200 \Rightarrow h = \frac{1200 - x^2}{4x}$$

$$\begin{aligned} V(x) &= x \cdot x \cdot h = x^2 h \\ &= x^2 \left( \frac{1200 - x^2}{4x} \right) = 300x - \frac{1}{4}x^3 \end{aligned}$$

$$\Rightarrow V'(x) = 300 - \frac{3}{4}x^2 = 0$$

$$\begin{aligned} \Rightarrow \frac{3}{4}x^2 &= 300 \Rightarrow x^2 = 400 \\ \Rightarrow x &= 20 \end{aligned}$$

$$V''(x) = -\frac{3}{2}x \Rightarrow V''(20) = -\frac{3}{2}(20) < 0$$

By the second derivative test,

$V(x)$  is maximum at  $x=20$ .

$$\begin{aligned} \Rightarrow V(20) &= (20)(300) - \frac{20^3}{4} \\ &= 6000 - 2000 = \boxed{4000 \text{ cm}^3} \end{aligned}$$

8. Use the **Fundamental Theorem of Calculus - Part I** to prove the **Fundamental Theorem of Calculus - Part II**. That is, if  $f(x)$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F(x)$  is any antiderivative of  $f(x)$ .

[10]

Proof:

Let  $g(x) = \int_a^x f(t) dt$ . Then by FTC-I,  $g'(x) = f(x)$ .  
Hence,  $g(x)$  is an antiderivative of  $f(x)$ . Since  $F(x)$  is also an antiderivative of  $f(x)$ , then  $F'(x) = g'(x) = f(x)$ .  
By corollary 4.2.7,  $F(x) = g(x) + C$ , where  $C$  is constant.  
Therefore,  $F(b) - F(a) = (g(b) + C) - (g(a) + C)$   
 $= g(b) - g(a)$   
 $= \int_a^b f(t) dt - \underbrace{\int_a^a f(t) dt}_{=0}$   
 $= \int_a^b f(t) dt$  ■

9. Evaluate the following integrals:

a.  $\int (8x^3 - 10e^{5x} + \sqrt{x} - x^{-1}) dx$

[10]

$$\begin{aligned} &= \frac{8x^4}{4} - 10 \frac{e^{5x}}{5} + \frac{x^{3/2}}{3/2} - \ln|x| + C \\ &= 2x^4 - 2e^{5x} + \frac{2}{3} x^{3/2} - \ln|x| + C \end{aligned}$$

$$\begin{aligned}
 \text{b. } \int_1^4 \frac{10 + 5x^2}{2\sqrt{x}} dx &= \int_1^4 \left( 5x^{-1/2} + \frac{5}{2} x^{3/2} \right) dx \\
 &= \left( \frac{5x^{1/2}}{1/2} + \frac{5}{2} \cdot \frac{x^{5/2}}{5/2} \right) \Big|_1^4 \\
 &= \left( 10\sqrt{x} + (\sqrt{x})^5 \right) \Big|_1^4 \\
 &= [10\sqrt{4} + (\sqrt{4})^5] - [10\sqrt{1} + (\sqrt{1})^5] \\
 &= (20 + 32) - 11 \\
 &= 41
 \end{aligned}
 \tag{10}$$

$$\begin{aligned}
 \text{c. } \int_0^{\pi/4} (1 + \tan \theta)^3 \sec^2 \theta d\theta &= \int_1^2 u^3 du = \frac{1}{4} u^4 \Big|_1^2 = \frac{1}{4} (2^4 - 1^4) \\
 \text{Let } u &= 1 + \tan \theta \\
 du &= \sec^2 \theta d\theta \\
 \text{when } \theta = 0 &\Rightarrow u = 1 + \tan(0) = 1 + 0 = 1 \\
 \theta = \pi/4 &\Rightarrow u = 1 + \tan(\pi/4) = 1 + 1 = 2 \\
 &= \frac{1}{4} (15) \\
 &= \frac{15}{4}
 \end{aligned}
 \tag{10}$$

10. A stone is thrown upward with an initial velocity of 64 ft/sec from the edge of a cliff 336 ft above the ground. If the only force acting on this ball is Earth's gravity ( $a = -32 \text{ ft/sec}^2$ ), answer the following questions:

- a. Determine the velocity of the ball,  $v(t)$ ,  $t$  seconds later. What is the velocity of the ball 1 second later? When will the ball attain its maximum height? [10]

$$\begin{aligned}
 \frac{dv}{dt} &= -32 \quad \rightarrow \quad v(t) = \int -32 dt = -32t + C_1 \\
 \text{when } t=0, \quad v(0) &= 64 \quad \rightarrow \quad v(t) = -32t + 64 \\
 \text{when } t=1 &\Rightarrow v(1) = -32(1) + 64 = 32 \text{ ft/sec.} \\
 \text{The stone will attain its maximum height when } v(t) &= 0 \\
 \Rightarrow -32t + 64 &= 0 \quad \Rightarrow \quad t = 2 \text{ sec.}
 \end{aligned}$$

- b. Determine the height function,  $h(t)$ ,  $t$  seconds later. When will the ball hit the ground and at what velocity will it hit the ground? [10]

$$\frac{dh}{dt} = v(t) \Rightarrow h(t) = \int (-32t + 64) dt$$

$$= -16t^2 + 64t + C_2$$

when  $t=0$ ,  $h(0) = 336 \Rightarrow h(t) = -16t^2 + 64t + 336 = 0$

$$-16(t^2 - 4t - 21) = 0$$

$$(t-7)(t+3) = 0$$

$$t=7 \quad | \quad t=-3$$

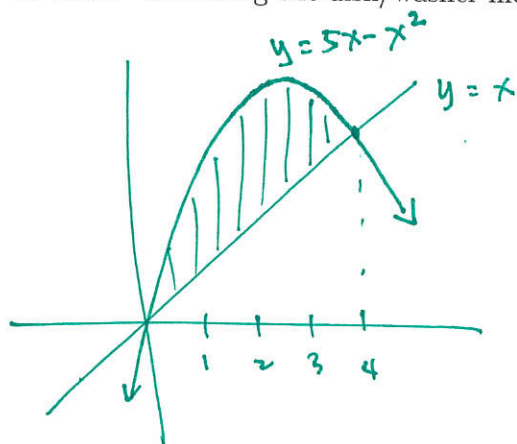
↖ Not possible

Hence, the stone will hit the ground 7 second later with a

velocity of  $v(7) = -32(7) + 64 = -160 \text{ ft/sec}$  ( $\approx -109.1 \text{ miles/hr}$ )

11. Calculate the volume of the solid of revolution when the region enclosed by the curves  $y = x$  and  $y = 5x - x^2$  is rotated about

- a. the  $x$ -axis using the disk/washer method.



$$x = 5x - x^2$$

$$\Rightarrow x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x=0 \quad | \quad x=4$$

$$y' = 5 - 2x = 0$$

$$x = \frac{5}{2} = 2.5$$

$$V = \int_0^4 \pi ( (5x - x^2)^2 - x^2 ) dx$$

$$= \pi \int_0^4 ( 25x^2 - 10x^3 + x^4 - x^2 ) dx$$

$$= \pi \left( \frac{24x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5} \right) \Big|_0^4$$

$$= \pi ( 8 \cdot 4^3 - 10(4^3) + \frac{4^5}{5} ) = \frac{384}{5} \pi$$

- b. the  $y$ -axis using the cylindrical shell method.

$$V = \int_0^4 2\pi x ( (5x - x^2) - x ) dx$$

$$= 2\pi \int_0^4 ( 4x^2 - x^3 ) dx$$

$$= 2\pi \left( \frac{4}{3}x^3 - \frac{1}{4}x^4 \right) \Big|_0^4$$

$$= 2\pi \left( \frac{4^4}{3} - 4^3 \right) = 2\pi \left( \frac{256}{3} - \frac{192}{3} \right)$$

$$= \frac{128\pi}{3}$$