Limits

- **Intuitive Definition of a Limit:** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say that the "limit of $f(x)$, as x approaches to a, equals L," if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

- **Definition of One-Sided Limits:** We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit of** $f(x)$**, as** x **approaches to** a [or the **limit of** $f(x)$ **as** x **approaches** a **from the left**] is equal L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a, and

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit of** $f(x)$**, as** x **approaches to** a **from the right** is equal L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *greater than* a.

- **Theorem 2.1:** $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ %vskip .2in

- **Intuitive Definition of an Infinite Limit:** Let $f(x)$ is defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we like) by taking x sufficiently close to a, but not equal to a, and

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we like) by taking x sufficiently close to a, but not equal to a.

- **Vertical Asymptotes:** The vertical line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{ccc} \lim_{x \to a} f(x) = \infty & \lim_{x \to a^-} f(x) = \infty & \lim_{x \to a^+} f(x) = \infty \\ \lim_{x \to a} f(x) = -\infty & \lim_{x \to a^-} f(x) = -\infty & \lim_{x \to a^+} f(x) = -\infty \end{array}$$

- **Limit Laws:** Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

  1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

  2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x)\neq 0$

- **Additional of Limit Laws:**

  1. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

  2. $\lim_{x\to a} c = c$

  3. $\lim_{x\to a} x = a$ where n is a positive integer

  4. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

  5. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. (If n is even, we assume $a > 0$)

  6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer. (If n is even, we assume $\lim_{x\to a} f(x) > 0$)

- **Direct Substitution Property:** If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

- **Theorem 2.2:** If $f(x)\leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then $\lim_{x\to a} f(x)\leq \lim_{x\to a} g(x)$.

- **The Squeeze Theorem:** If $f(x)\leq g(x)\leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

- **Precise Definition of a Limit:** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches to a is L**, and we write

$$\lim_{x\to a} f(x) = L$$

  if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad |f(x)-L| < \epsilon.$$

- **Precise Definition of an Infinite Limit:** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

  means that for every positive number M there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad f(x) > M$$

  Also

$$\lim_{x\to a} f(x) = -\infty$$

  means that for every negative number N there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad f(x) < N$$