Integrals

- **Definite Integral:** If $f(x)$ is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(=a), x_1, x_2, \ldots, x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the *definite integral of $f(x)$ from a to b* is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Theorem 5.2.1:** If $f(x)$ is continuous on $[a, b]$, or if $f(x)$ has only a finite number of jump discontinuities, then $f(x)$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

- **Properties of Definite Integral:**

  1. $\int_a^a f(x)\,dx = 0$
  2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
  3. $\int_a^b c\,dx = c(b-a)$, where c is any constant
  4. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  5. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
  6. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
  7. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  8. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
  9. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
  10. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

- **Fundamental Theorem of Calculus, Part I:** If $f(x)$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

  is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

- **Fundamental Theorem of Calculus, Part II:** If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

  where $F(x)$ is any antiderivative of $f(x)$.