Covariance and Correlation

- **Expected Value.** Let X and Y be jointly distributed rv's with pmf $p(x, y)$ or pdf $f(x, y)$ according to whether the variables are discrete or continuous. Then the expected value of a function $h(X, Y)$, denoted by $E[h(X, Y)]$ or $\mu_{h(X,Y)}$, is given by

$$E[h(X,Y)] = \begin{cases} \sum_x \sum_y h(x,y)p(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)\,dxdy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

Example 1: Consider the random variables given in example 5.1. Find $E(X + Y)$.

Example 2: Consider the random variables given in example 5.5. Find $E(X + Y)$.

- **Covariance.** The *covariance* between two random variables X and Y is

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E(XY) - \mu_X\mu_Y \\ &= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)p(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)f(x,y)\,dxdy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases} \end{aligned}$$

Note: $Cov(X, X) = Var(X)$.
Example 3: Consider the random variables given in example 5.1. Find $Cov(X, Y)$.

Example 4: Consider the random variables given in example 5.5. Find $Cov(X, Y)$.

- **Correlation.** The *correlation coefficient* of X and Y, denoted by $Corr(X, Y)$, $\rho_{X,Y}$, or just ρ, is defined by

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Example 5: Consider the random variables given in example 5.1. Find $Corr(X, Y)$.

Example 6: Consider the random variables given in example 5.5. Find $Corr(X, Y)$.

- **Propositions.**

  1. If a and c are either both positive or both negative, then $Corr(aX + b, cY + d) = Corr(X, Y)$.

  2. For any two rv's X and Y, $-1 \leq Corr(X, Y) \leq 1$.

  3. If X and Y are independent, then $\rho = 0$, but $\rho = 0$ does not imply independence.

  4. $\rho = 1$ or $\rho = -1$ iff $Y = aX + b$ for some numbers a and b with $a \neq 0$.

- **Homework problems:**
  Section 5.2: pp. 211-212; # 22, 27, 28, 33, 35.