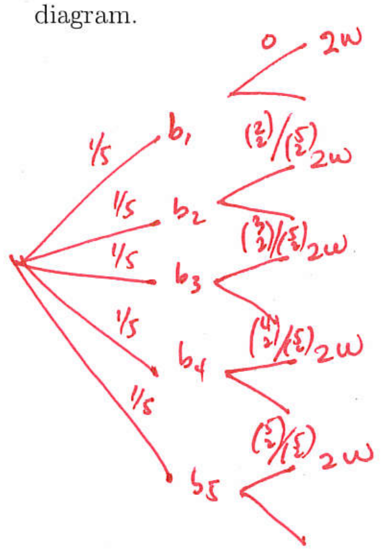
NAME ______________________ LONG EXAM I

MATH 441 - MATHEMATICAL STATISTICS I SEPT. 26, 2007

Instructions: *Please include all relevant work to get full credit. Encircle your final answers.*

1. Five identical bowls are labeled 1, 2, 3, 4, and 5. Bowl i contains i white and $5-i$ black balls, with $i=1,2,\ldots,5$. A bowl is randomly selected and two balls are randomly drawn (without replacement) from the contents of the bowl.

a. What is the probability that both balls selected are white? Construct the appropriate tree diagram. [8]



P(both are white)

$$= \left[0 + \frac{\binom{2}{2}}{\binom{5}{2}} + \frac{\binom{3}{2}}{\binom{5}{2}} + \frac{\binom{4}{2}}{\binom{5}{2}} + \frac{\binom{5}{2}}{\binom{5}{2}}\right]\cdot\frac{1}{5}$$

$$= \frac{1+3+6+10}{50} = \frac{20}{50} = \frac{2}{5} = .4$$

b. Given that both balls selected are white, what is the probability that bowl 3 was selected? [4]

$$= \frac{3/50}{20/50} = \frac{3}{20}$$

2. If events A and B are independent, show that A^c and B^c are also independent.
[Hint: $(A^c \cap B^c) = (A \cup B)^c \Rightarrow P(A^c \cap B^c) = 1 - P(A \cup B)$]. [6]

$P(A^c \cap B^c) = 1 - P(A \cup B)$ by De Morgan's Law + Thm 2.7

$= 1 - [P(A) + P(B) - P(A \cap B)]$ by Thm 2.6

$= 1 - P(A) - P(B) + P(A)P(B)$ since A + B are Ind.

$= (1-P(A))(1-P(B))$

$= P(A^c)P(B^c)$ ∎

3. Let Y be a random variable with probability mass function given by

$$P(Y = y) = k\binom{5}{y}, \qquad y = 0, 1, 2, 3.$$

a. Find the value of k. [6]

y	0	1	2	3
$p(y)$	$k\binom{5}{0}$	$k\binom{5}{1}$	$k\binom{5}{2}$	$k\binom{5}{3}$

$\Rightarrow \quad k + 5k + 10k + 10k = 1$

$26k = 1$

$k = \frac{1}{26}$

b. Find mean of Y. [6]

y	0	1	2	3
$p(y)$	$\frac{1}{26}$	$5/26$	$10/26$	$10/26$

$\Rightarrow E(y) = 0 + \frac{5}{26} + \frac{20}{26} + \frac{30}{26} = \frac{55}{26} \quad 2.115$

c. Find $P(Y = 1|Y < 3)$. $= \dfrac{P(Y=1,\ Y<3)}{P(Y<3)} = \dfrac{P(Y=1)}{P(Y=3)} = \dfrac{5/26}{16/26}$ [6]

$= \frac{5}{16} \quad .3125$

4. Suppose Y has the poisson distribution with mean $\lambda = 16$.

a. Find $P(\cancel{10}\ 10 \le Y \le \cancel{16}\ 22)$. $= P(Y \le 22) - P(Y \le 9) = 0.942 - 0.043$ [6]

$\approx$ ~~.819 − .193~~ $= 0.899$

$=$ ~~0.619~~

b. What does Tchebysheff's Theorem says about $P(10 \le Y \le \cancel{16}^{22})$? [6]

Since $Y \sim pois(\lambda = 16) \Rightarrow E(Y) = 16$ & $V(Y) = 16$ or $SD(Y) = 4$

$$\Rightarrow P(10 \le Y \le 22) = P(-6 \le Y - 16 \le 6)$$
$$= P(|Y-16| \le 6)$$
$$= P(|Y-16| \le \tfrac{3}{2}(4)) \ge 1 - \frac{1}{(3/2)^2} = 1 - \frac{4}{9} = \frac{5}{9}$$

5. Geologists estimate that the success probability that an exploratory oil well drilled in a particular region will strike oil is 0.20. If this estimate is accurate, what is the probability that

a. the first strike comes on the third well drilled? $(.8)^2(.2) = .128$ [3]

b. the third strike comes on the fifth well drilled? $\binom{4}{2}(.8)^2(.2)^3 = .03072$ [3]

c. What 3 assumptions did you make to obtain the answers to (a) and (b)? [3]

i. Each trial has only 2 possible outcomes—will strike oil or not.

ii. The probability of successfully striking oil (p) remains the same from trial to trial.

iii. The trials are independent.

d. Find the mean number of wells that must be drilled if the company wants to set up three producing wells. [3]

Since $Y \sim Nbin(r = 3, p = .2)$

$$\Rightarrow E(Y) = \frac{3}{.2} = 15$$

6. In the previous question, suppose that they had their third oil strike on the fifth well drilled. Determine the value of p that maximizes the probability of observing this event. [8]

$$\Rightarrow L(p) = \binom{4}{2} \cdot (p^3)(1-p)^2$$

$$l(p) = \ln(L) = \ln\binom{4}{2} + 3\ln p + 2\ln(1-p)$$

$$\frac{dl}{dp} = \frac{3}{p} - \frac{2}{1-p} = 0 \Rightarrow \frac{3-3p-2p}{p(1-p)} = 0$$

$$\Rightarrow 3 - 5p = 0$$

$$\Rightarrow p = \frac{3}{5} = .60$$

$$\frac{d^2 l}{dp^2} = -\frac{3}{p^2} - \frac{2}{(1-p)^2} < 0$$

$\Rightarrow l(p)$ is maximum at $p = .60$

Therefore, $L(p)$ is max. at $p = .60$

7. Suppose $Y \sim bin(n, p)$.

a. Use the definition of expected value to show that $E(Y) = np$. [6]

$$E(Y) = \sum_{y=0}^{n} y \binom{n}{y} p^{y}(1-p)^{n-y}$$

$$= \sum_{y=1}^{n} y \cdot \frac{n!}{y!(n-y)!} p^{y}(1-p)^{n-y}$$

$$= np \underbrace{\sum_{y=1}^{n} \frac{(n-1)!}{(y-1)!(n-y)!} p^{(y-1)}(1-p)^{(n-1)-(y-1)}}_{=1}$$

$$= np \quad \blacksquare$$

b. Use the fact that $V(Y) = E(\cancel{X}(\cancel{X}-1)) + E(\cancel{X}) - [E(\cancel{X})]^2$ to show that $V(Y) = np(1-p)$. [6]

$$E(X(X-1)) = \sum_{y=0}^{n} y(y-1) \frac{n!}{y!(n-y)!} p^{y}(1-p)^{n-y}$$

$$= p^2 n(n-1) \underbrace{\sum_{y=2}^{n} \frac{(n-2)!}{(y-2)!(n-y)!} p^{y-2}(1-p)^{(n-2)-(y-2)}}_{=1}$$

$$= p^2(n^2 - n)$$

$$\Rightarrow V(Y) = \cancel{n^2p^2} - np^2 + (np) - \cancel{(np)^2}$$

$$= np(1-p) \quad \blacksquare$$

8. The moment generating function of a random variable Y is given by $m(t) = (1 - 5t)^{-4}$.

a. Find the mean of Y. [5]

$$E(Y) = m'(0) = -4(1-5t)^{-5}(-5)\Big|_{t=0}$$

$$= 20.$$

b. Find the variance of Y. [7]

$$E(Y^2) = m''(0) = 20(-5)(1-5t)^{-6}(-5)\Big|_{t=0}$$

$$= 500$$

$$\Rightarrow V(Y) = E(Y^2) - E(Y)^2 = 500 - (20)^2 = 100.$$

c. Find $E(Y^2 - 10Y - 100)$. [4]

$$= E(Y^2) - 10E(Y) - E(100)$$

$$= 500 - 10(20) - 100$$

$$= 200.$$

9. If Y has a geometric distribution with probability of success p. show that the moment-generating function for Y is $m(t) = \dfrac{pe^t}{1 - qe^t}$, where $q = 1 - p$. [8]

Solution:

$$m(t) = E(e^{tY}) = \sum_{y=1}^{\infty} e^{ty}\, p(1-p)^{y-1}$$

$$= pe^t \sum_{y=1}^{\infty} (e^t)^{y-1}(1-p)^{y-1}$$

$$= \frac{pe^t}{1-qe^t} \underbrace{\sum_{y=1}^{\infty} (qe^t)^{y-1}(1-qe^t)}_{=1}$$

$$= \frac{pe^t}{1-qe^t} \quad \blacksquare$$