NAME Key — OCTOBER 19, 2007

MATH 441 - MATHEMATICAL STATISTICS I — LONG EXAM II

Instructions: *Please include all relevant work to get full credit. Write your solutions using proper notations and do not forget to include the support set whenever you give a density function.*

1. Let $Y \sim Gamma(\alpha, \beta)$. That is, $f(y) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}$, with $\alpha, \beta > 0$ for $y > 0$.

a. Use the above density function to show that $E(Y) = \alpha\beta$ and $V(Y) = \alpha\beta^2$.
[Hint: Use the fact that $\int_0^\infty y^{\alpha-1} e^{-y/\beta}\, dy = \beta^\alpha \Gamma(\alpha)$]. [10]

b. Show that the moment-generating function of Y is $m(t) = (1 - \beta t)^{-\alpha}$. [8]

c. Use this moment-generating function to show that $E(Y) = \alpha\beta$ and $V(Y) = \alpha\beta^2$. [10]

2. Let Y_1, Y_2, and Y_3 be random variables with the following expected values:

$E(Y_1) = 2$	$E(Y_2) = -1$	$E(Y_3) = -2$
$V(Y_1) = 1$	$V(Y_2) = 2$	$V(Y_3) = 4$
$E(Y_1Y_2) = -2$	$E(Y_1Y_3) = -3$	$E(Y_2Y_3) = 4$

Let $U_1 = 2Y_1 - 3Y_2 + Y_3$ and $U_2 = 3Y_1 + 2Y_2$.

a. Find mean of U_1. [4]

b. Find the variance of U_1. [10]

c. Find $Cov(U_1, U_2)$. [8]

d. Are the random variables Y_2 and Y_3 independent? Explain how you arrived at your answer. [2]

3. Suppose that the random variables X and Y have joint probability density function

$$f(x, y) = 6x^2 y, \qquad 0 \le x \le y, \text{ and } x + y \le 2,$$

and 0 elsewhere. Find the marginal density function of X. [10]

4. Suppose Y_1 is a random variable with density function $f_{Y_1}(y_1) = 5y_1^4, 0 < y_1 < 1$, and the conditional density of Y_2 given Y_1 is $f_{Y_2|Y_1}(y_2|y_1) = \dfrac{2y_2}{y_1}$, $0 < y_2 < y_1 < 1$.

a. Find $P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2})$. [6]

b. Find $E[Y_2|Y_1]$. [8]

c. Find $E[Y_2]$. [6]

d. Set up the integrals to find the following:

i. Marginal density of Y_2. [5]

ii. Find $P(Y_2 > \frac{1}{4}, Y_1 > \frac{1}{2})$. [5]

5. Let Y_1 be an exponentially distributed random variable with mean 1 and let Y_2 be a random variable following a gamma distribution with $\alpha = 2$ and $\beta = 1$. If the joint density function of Y_1 and Y_2 is given by $f(y_1, y_2) = e^{-y_2}$, $0 < y_1 < y_2 < \infty$, determine the variance of $U = 10Y_1 - 5Y_2$. [10]

Math 441 - Exam #2

#1

a) $E(Y) = \int_0^\infty y \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^\alpha e^{-y/\beta} dy$

$= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} = \alpha\beta$ ■

$V(Y) = E(Y^2) - E(Y)^2$

$E(Y^2) = \int_0^\infty y^2 \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta} dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-y/\beta} dy$

$= \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha \Gamma(\alpha)} = \beta^2 \alpha(\alpha+1)$

$\Rightarrow V(Y) = \beta^2\alpha(\alpha+1) - (\alpha\beta)^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$ ■

b) $m(t) = E[e^{tY}] = \int_0^\infty e^{ty} \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta} dy$

$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty y^{\alpha-1} e^{-y\left(\frac{1}{\beta}-t\right)} dy}_{=\left[\left(\frac{1}{\beta}-t\right)^{-1}\right]^\alpha \Gamma(\alpha)} = \frac{\left(\frac{1-t\beta}{\beta}\right)^{-\alpha} \cdot \cancel{\Gamma(\alpha)}}{\beta^\alpha \cancel{\Gamma(\alpha)}}$

$= (1-t\beta)^{-\alpha}$ ■

c) $E(Y) = \frac{d}{dt}(m(t))\Big|_{t=0} = -\alpha(1-t\beta)^{-\alpha-1}(-\beta)\Big|_{t=0} = \alpha\beta$ ■

$E(Y^2) = \frac{d^2}{dt} m(t)\Big|_{t=0} = \alpha\beta(-\alpha-1)(1-t\beta)^{-\alpha-2}(-\beta)\Big|_{t=0}$

$= \beta^2\alpha(\alpha+1)$

$V(Y) = \beta^2\alpha^2 + \beta^2\alpha - (\alpha\beta)^2 = \alpha\beta^2$ ■

#2 a) $E[2Y_1 - 3Y_2 + Y_3] = 2(2) - 3(-1) + (-2) = 4 + 3 - 2 = 5$

b) $V(2Y_1 - 3Y_2 + Y_3) = 4Var(Y_1) + 9Var(Y_2) + Var(Y_3)$

$+ 2[2(-3)Cov(Y_1, Y_2) + 2(1)Cov(Y_1, Y_3) + (-3)(1)Cov(Y_2, Y_3)]$

$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = -2 - (2)(-1) = 0$

$Cov(Y_1, Y_3) = -3 - (2)(-2) = 1$

$Cov(Y_2, Y_3) = 4 - (-1)(-2) = 2$

$\Rightarrow V(U_1) = 4(1) + 9(2) + (4) + 2[-6(0) + 2(1) - 3(2)]$

$= 4 + 18 + 4 + 0 + 4 - 12 = 18$

c) $Cov(U_1, U_2) = 6\,Cov(Y_1, Y_1) + 4\,Cov(Y_1, Y_2) + (-3)(3)Cov(Y_2, Y_1)$

$+ (-3)(2)Cov(Y_2, Y_2) + 3\,Cov(Y_3, Y_1) + 2\,Cov(Y_3, Y_2)$

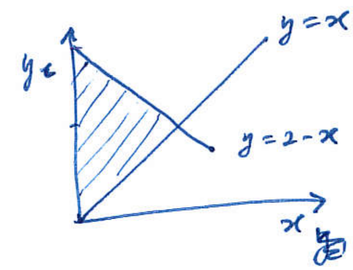
$= 6(1) + 4(0) - 9(0) - 6(2) + 3(1) + 2(2)$

$= 6 - 12 + 3 + 4 = 1$

d) No, because $Cov(Y_2, Y_3) = 2 \neq 0$.

#3

$f(x,y) = 6x^2y$, $0 \le x \le y$ and $x + y \le 2$



$f_X(x) = \int_x^{2-x} 6x^2y\,dy = 6x^2\left(\frac{1}{2}y^2\Big|_x^{2-x}\right)$

$= 3x^2((2-x)^2 - x^2)$

$= 3x^2(4 - 4x)$, $0 \le x \le 1$

#4 a) $P(Y_2 > \frac{1}{4} \mid Y_1 = \frac{1}{2}) = \int_{1/4}^{1/2} 4y_2\,dy_2 = 2y_2^2\Big|_{1/4}^{1/2} = 2\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{8}$

b) $E[Y_2 \mid Y_1] = \int_0^{y_1} y_2\left(\frac{2y_2}{y_1}\right)dy_2 = \frac{2}{y_1}\left(\frac{1}{3}y_2^3\Big|_0^{y_1}\right)$

$= \frac{2}{3}y_1^2 \quad , \; 0 < y < 1 .$

c) $E[Y_2] = E_{Y_1}[E(Y_2 \mid Y_1)] = E_{Y_1}\left[\frac{2}{3}Y_1^2\right] = \int_0^1 \frac{2}{3}y_1^2(5y_1^4)\,dy_1$

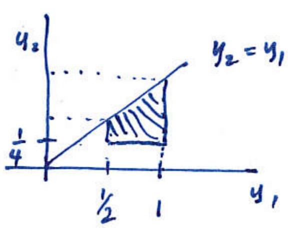
$= \int_0^1 \frac{10}{3}y_1^6\,dy_1 = \frac{10}{3}\left(\frac{1}{7}y_1^7\Big|_0^1\right) = \frac{10}{21}$

d) i) $f_{Y_2}(y_2) = \int_{y_2}^1 10y_1^3 y_2\,dy_1$, since $f(y_1,y_2) = f_{Y_2|Y_1}(y_2 \mid y_1)\, f_{Y_1}(y_1)$

$= \frac{2y_2}{y_1}(5y_1^4)$

$= 10y_1^3 y_2 \quad , \; 0 < y_2 < y_1 < 1$

ii) $P(Y_2 > \frac{1}{4}, Y_1 > \frac{1}{2}) = \int_{1/2}^1 \int_{1/4}^{y_1} 10y_1^3 y_2\,dy_2\,dy_1$



#5

$Y_1 \sim \exp(1) \Rightarrow f(y_1) = e^{-y_1}, \; y_1 > 0 \rightarrow V(Y_1) = 1$ and $E(Y_1) = 1$

$Y_2 \sim G(\alpha = 2, \beta = 1) \Rightarrow f(y_2) = y_2 e^{-y_2}, \; y_2 > 0 \rightarrow V(Y_2) = 2$ and $E(Y_2) = 2$

and $f(y_1, y_2) = e^{-y_2} \quad , \; 0 < y_1 < y_2 < \infty .$

By theorem 5.12, $V(10Y_1 - 5Y_2) = 100\,Var(Y_1) + 25\,Var(Y_2) + 2(10)(-5)\,Cov(Y_1, Y_2)$

with $Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = E(Y_1Y_2) - 1(2)$

but $E(Y_1Y_2) = \int_0^\infty \int_0^{y_2} y_1 y_2 e^{-y_2}\,dy_1\,dy_2 = \int_0^\infty y_2 e^{-y_2}\left(\frac{y_1^2}{2}\Big|_0^{y_2}\right)dy_2 = \frac{1}{2}\underbrace{\int_0^\infty y_2^3 e^{-y_2}\,dy_2}_{=\Gamma(4) = 3!}$

$= \frac{1}{2}(6) = 3$

hence, $Cov(Y_1, Y_2) = 3 - 2 = 1$

Therefore, $V(10Y_1 - 5Y_2) = 100(1) + 25(2) - 100(1) \boxed{= 50}$.