NAME ______________________ OCTOBER 19, 2007

MATH 441 - MATHEMATICAL STATISTICS I LONG EXAM II

Instructions: *Please include all relevant work to get full credit. Write your solutions using proper notations and do not forget to include the support set whenever you give a density function.*

1. Let $Y \sim Gamma(\alpha, \beta)$. That is, $f(y) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}$, with $\alpha, \beta > 0$ for $y > 0$.

a. Use the above density function to show that $E(Y) = \alpha\beta$ and $V(Y) = \alpha\beta^2$.
[Hint: Use the fact that $\int_0^\infty y^{\alpha-1} e^{-y/\beta}\, dy = \beta^\alpha \Gamma(\alpha)$]. [10]

b. Show that the moment-generating function of Y is $m(t) = (1 - \beta t)^{-\alpha}$. [8]

c. Use this moment-generating function to show that $E(Y) = \alpha\beta$ and $V(Y) = \alpha\beta^2$. [10]

2. Let Y_1, Y_2, and Y_3 be random variables with the following expected values:

$E(Y_1) = 2$	$E(Y_2) = -1$	$E(Y_3) = -2$
$V(Y_1) = 1$	$V(Y_2) = 2$	$V(Y_3) = 4$
$E(Y_1Y_2) = -2$	$E(Y_1Y_3) = -3$	$E(Y_2Y_3) = 4$

Let $U_1 = 2Y_1 - 3Y_2 + Y_3$ and $U_2 = 3Y_1 + 2Y_2$.

a. Find mean of U_1. [4]

b. Find the variance of U_1. [10]

c. Find $Cov(U_1, U_2)$. [8]

d. Are the random variables Y_2 and Y_3 independent? Explain how you arrived at your answer. [2]

3. Suppose that the random variables X and Y have joint probability density function

$$f(x, y) = 6x^2y, \qquad 0 \le x \le y, \text{ and } x + y \le 2,$$

and 0 elsewhere. Find the marginal density function of X. [10]

4. Suppose Y_1 is a random variable with density function $f_{Y_1}(y_1) = 5y_1^4, 0 < y_1 < 1$, and the conditional density of Y_2 given Y_1 is $f_{Y_2|Y_1}(y_2|y_1) = \dfrac{2y_2}{y_1}$, $0 < y_2 < y_1 < 1$.

a. Find $P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2})$. [6]

b. Find $E[Y_2|Y_1]$. [8]

c. Find $E[Y_2]$. [6]

d. Set up the integrals to find the following:

i. Marginal density of Y_2. [5]

ii. Find $P(Y_2 > \frac{1}{4}, Y_1 > \frac{1}{2})$. [5]

5. Let Y_1 be an exponentially distributed random variable with mean 1 and let Y_2 be a random variable following a gamma distribution with $\alpha = 2$ and $\beta = 1$. If the joint density function of Y_1 and Y_2 is given by $f(y_1, y_2) = e^{-y_2}$, $0 < y_1 < y_2 < \infty$, determine the variance of $U = 10Y_1 - 5Y_2$. [10]