NAME ______________________ NOVEMBER 9, 2007

MATH 441 - MATHEMATICAL STATISTICS I LONG EXAM III

Instructions: *Please include all relevant work to get full credit. Write your solutions neatly and using proper notations. Do not forget to include the support set whenever you give a density function.*

1. Let $Y \sim N(\mu, \sigma^2)$. That is, $f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$.

a. Use the Transformation method to show that $Z = (Y - \mu)/\sigma$ is standard normal. [10]

b. Use the Distribution method to show that $X = Z^2 = (Y-\mu)^2/\sigma^2$ has the chi-square distribution with 1 degree of freedom. Note: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. [12]

a) $z = \frac{Y-\mu}{\sigma} \Rightarrow Y = \mu + z\sigma \Rightarrow \frac{dy}{dz} = \sigma$

$$f_Z(z) = f_Y(y)\frac{dy}{dz} = \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{1}{2\sigma^2}(\cancel{\mu}+z\sigma-\cancel{\mu})^2} \cdot \cancel{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty.$$

This is the density function of $N(0,1)$.

b) $$\chi^2_{(1)} \equiv \text{Gamma}\left(\alpha = \tfrac{1}{2}, \beta = 2\right) = \frac{1}{\Gamma(\frac{1}{2})2^{1/2}} x^{\frac{1}{2}-1} e^{-x/2}, \quad x > 0$$

$$= \frac{1}{\sqrt{\pi}\sqrt{2}\sqrt{x}} e^{-x/2}, \quad x > 0.$$

If $X = Z^2$ and $Z \sim N(0,1)$

Then, $F_X(x) = P(X \le x) = P(Z^2 \le x) = P(-\sqrt{x} \le Z \le \sqrt{x})$

$$= F_Z(\sqrt{x}) - F_Z(-\sqrt{x})$$

$$\Rightarrow f_X(x) = f_Z(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) - f_Z(\sqrt{x})\left(\frac{-1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}}\left(f_Z(\sqrt{x}) + f_Z(-\sqrt{x})\right)$$

$$= \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{x}} e^{-\frac{1}{2}x} \equiv \text{Gamma}\left(\alpha = \tfrac{1}{2}, \beta = 2\right) \equiv \chi^2_{(1)}$$

2. Let $Y_1, Y_2, \ldots, Y_n$ be independent normal random variables with known mean μ and unknown variance σ^2. Use the result of question 1 part (b) to show that $V = \dfrac{nS^2}{\sigma^2}$, where $S^2 = \dfrac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)^2$, follows the chi-square distribution with n degrees of freedom. Note: The moment-generating function of $Gamma(\alpha, \beta)$ is $(1 - \beta t)^{-\alpha}$. [12]

Solution:

$$V = \frac{nS^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(Y_i - \mu)^2}{\sigma} = \sum_{i=1}^{n}\left(\frac{Y_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} Z_i^2$$

Since the Y_i's are independent, then so are the Z_i's.

From question 1 part (b), we know that $Z_i^2 \sim Gamma(\alpha = \frac{1}{2}, \beta = 2)$.

Hence, the moment-generating function of Z_i^2 is $(1-2t)^{-1/2}$.

Because the Z_i^2's are independent, then the moment-generating function of $V = \sum_{i=1}^{n} Z_i^2$ is given by

$$m_V(t) = \prod_{i=1}^{n}(1-2t)^{-1/2} = (1-2t)^{-n/2}.$$

Therefore, $V \sim Gamma\left(\alpha = \frac{n}{2}, \beta = 2\right) \equiv \chi^2_{(n)}$. $\blacksquare$

3. Suppose that $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent random variables. Let $U_1 = \dfrac{Y_1}{Y_1 + Y_2}$ and $U_2 = Y_1 + Y_2$.

a. Derive the joint density function of U_1 and U_2. Don't forget to specify the support set. [12]

b. Show that the marginal distribution of U_2 is $Gamma(\alpha_1 + \alpha_2, \beta)$. [12]

c. Show that the marginal distribution of U_1 is $Beta(\alpha_1, \alpha_2)$. [12]

a) $$f_{Y_1}(y_1) = \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} y_1^{\alpha_1 - 1} e^{-y_1/\beta} \quad \text{and} \quad f_{Y_2}(y_2) = \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} y_2^{\alpha_2 - 1} e^{-y_2/\beta}, \quad y_1 > 0,\ y_2 > 0$$

$$\Rightarrow f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} e^{-(y_1+y_2)/\beta}$$

Since $U_1 = \dfrac{Y_1}{Y_1+Y_2}$ and $U_2 = Y_1 + Y_2$, $\Rightarrow$ $U_2 > 0$ and $0 < U_1 < 1$

then $Y_1 = U_1 U_2$ and $Y_2 = U_2 - U_1 U_2 = U_2(1 - U_1)$

$$\Rightarrow J = \begin{vmatrix} u_2 & u_1 \\ -u_2 & 1-u_1 \end{vmatrix} = u_2(1-u_1) + u_1 u_2 = u_2$$

Hence,

$$f_{U_1,U_2}(u_1,u_2) = \frac{u_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} (u_1 u_2)^{\alpha_1 - 1} (u_2(1-u_1))^{\alpha_2 - 1} e^{-u_2/\beta}$$

$$= \frac{u_2^{\alpha_1+\alpha_2-1} u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} e^{-u_2/\beta}, \quad 0 < u_1 < 1,\ u_2 > 0.$$

b) $$f_{U_2}(u_2) = \frac{u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}}{\cancel{\Gamma(\alpha_1)}\cancel{\Gamma(\alpha_2)}\beta^{\alpha_1+\alpha_2}} \underbrace{\int_0^1 u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}\,du_1}_{= \frac{\cancel{\Gamma(\alpha_1)}\cancel{\Gamma(\alpha_2)}}{\Gamma(\alpha_1+\alpha_2)}}$$

$$= \frac{u_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} e^{-u_2/\beta}, \quad u_2 > 0$$

This is the density function of $Gamma(\alpha_1+\alpha_2, \beta)$.

c) $$f_{U_1}(u_1) = \frac{u_1^{\alpha_1 - 1}(1-u_1)^{\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cancel{\beta^{\alpha_1+\alpha_2}}} \underbrace{\int_0^\infty u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}\, du_2}_{= \Gamma(\alpha_1+\alpha_2)\cancel{\beta^{\alpha_1+\alpha_2}}}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}, \quad 0 < u_1 < 1$$

This is the density function of Beta(α_1, α_2).

Alternative solutions to part (b) and (c).

(b) Since $Y_1 \sim$ Gamma $(\alpha_1, \beta) \Rightarrow m_{Y_1}(t) = (1-\beta t)^{-\alpha_1}$

and $Y_2 \sim$ Gamma $(\alpha_2, \beta) \Rightarrow m_{Y_2}(t) = (1-\beta t)^{-\alpha_2}$

Because Y_1 and Y_2 are independent, then the moment-generating function of $U_2 = Y_1 + Y_2$ is given by

$$m_{U_2}(t) = (1-\beta t)^{-\alpha_1}(1-\beta)^{-\alpha_2} = (1-\beta t)^{-(\alpha_1+\alpha_2)}.$$

Therefore, $U_2 \sim$ Gamma $(\alpha_1 + \alpha_2, \beta)$.

(c) From the joint density of U_1 and U_2 in part (a), we can see that U_1 and U_2 are independent because it can be factored as $g(u_1) \cdot h(u_2)$.

Since $U_2 \sim$ Gamma $(\alpha_1+\alpha_2, \beta) \Rightarrow f_{U_2}(u_2) = \dfrac{u_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} e^{-u_2/\beta}$

Then $$f_{U_1}(u_1) = \frac{f_{U_1,U_2}(u_1,u_2)}{f_{U_2}(u_2)} = \frac{\dfrac{\cancel{u_2^{\alpha_1+\alpha_2-1}}\, u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}\cancel{e^{-u_2/\beta}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cancel{\beta^{\alpha_1+\alpha_2}}}}{\dfrac{\cancel{u_2^{\alpha_1+\alpha_2-1}}}{\Gamma(\alpha_1+\alpha_2)\cancel{\beta^{\alpha_1+\alpha_2}}}\cancel{e^{-u_2/\beta}}}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u_1^{\alpha_1-1}(1-u_1)^{\alpha_2-1}$$

$0 < u_1 < 1$

Therefore, $U_1 \sim$ Beta (α_1, α_2)

4. Let Y_1, Y_2, Y_3, Y_4, and Y_5 be independent and identically distributed random variables with pdf, $f(y) = 3(1-y)^2, 0 < y < 1$.

a. Find $P(Y_{(1)} < 0.1)$. [12]

b. Find the joint probability density function of $Y_{(1)}$ and $Y_{(5)}$. [10]

c. Sketch the region $Y_{(5)} - Y_{(1)} > 0.8$ on the support set and then, <u>set up</u> the integrals to compute $P(Y_{(5)} - Y_{(1)} > 0.8)$. [10]

Since $f_Y(y) = 3(1-y)^2, \; 0<y<1 \Rightarrow F_Y(y) = 1-(1-y)^3, \; 0<y<1.$

a) $Y_{(1)} \sim n[1-F_Y(y)]^{n-1} f_Y(y) = 5\left[1-[1-(1-y)^3]\right]^4 3(1-y)^2$

$= 15(1-y)^{14}, \; 0<y<1.$

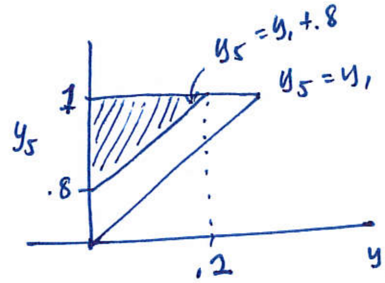
$\Rightarrow P(Y_{(1)} < 0.1) = \int_0^{.1} 15(1-y)^{14}\,dy = -(1-y)^{15}\Big|_0^{0.1} = 1-(.9)^5 \approx .7941$

b) $f_{(j)(k)}(y_j)(y_k) = \dfrac{n!}{(j-1)!\,(k-1-j)!\,(n-k)!}[F_Y(y_j)]^{j-1}[F_Y(y_k)-F_Y(y_j)]^{k-1-j}[1-F(y_k)]^{n-k} f(y_j)f(y_k)$

$\Rightarrow f_{(1)(5)}(y_1)(y_5) = \dfrac{5!}{3!}\left[(1-(1-y_5)^3) - (1-(1-y_1)^3)\right]^{5-2} 3(1-y_1)^2\, 3(1-y_5)^2$

$= 180\,(1-y_1)^2(1-y_2)^2\left[(1-y_1)^3-(1-y_5)^3\right]^3, \quad 0<y_1<y_5<1$

c) $Y_{(5)} - Y_{(1)} > .8 \Rightarrow$



$P(Y_{(5)} - Y_{(1)} > .8) = \int_0^{.2}\int_{y_1+.8}^{1} 180(1-y_1)^2(1-y_2)^2\left[(1-y_1)^3-(1-y_5)^3\right]^3 dy_5\,dy_1$

$= \int_{.8}^{1}\int_0^{y_5-.8} 180(1-y_1)^2(1-y_2)^2\left[(1-y_1)^3-(1-y_5)^3\right]^3 dy_1\,dy_5$