Name ______________________________ November 9, 2007

Math 441 - Mathematical Statistics I Long Exam III

Instructions: *Please include all relevant work to get full credit. Write your solutions neatly and using proper notations. Do not forget to include the support set whenever you give a density function.*

1. Let $Y \sim N(\mu, \sigma^2)$. That is, $f(y) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$.

- **a.** Use the Transformation method to show that $Z = (Y - \mu)/\sigma$ is standard normal. [10]
- **b.** Use the Distribution method to show that $X = Z^2 = (Y - \mu)^2/\sigma^2$ has the chi-square distribution with 1 degree of freedom. Note: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. [12]

2. Let $Y_1, Y_2, \ldots, Y_n$ be independent normal random variables with known mean μ and unknown variance σ^2. Use the result of question 1 part (b) to show that $V = \dfrac{nS^2}{\sigma^2}$, where $S^2 = \dfrac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)^2$, follows the chi-square distribution with n degrees of freedom. Note: The moment-generating function of $Gamma(\alpha, \beta)$ is $(1 - \beta t)^{-\alpha}$. [12]

3. Suppose that $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent random variables. Let $U_1 = \dfrac{Y_1}{Y_1 + Y_2}$ and $U_2 = Y_1 + Y_2$.

a. Derive the joint density function of U_1 and U_2. Don't forget to specify the support set. [12]

b. Show that the marginal distribution of U_2 is $Gamma(\alpha_1 + \alpha_2, \beta)$. [12]

c. Show that the marginal distribution of U_1 is $Beta(\alpha_1, \alpha_2)$. [12]

4. Let Y_1, Y_2, Y_3, Y_4, and Y_5 be independent and identically distributed random variables with pdf, $f(y) = 3(1-y)^2, 0 < y < 1$.

a. Find $P(Y_{(1)} < 0.1)$. [12]

b. Find the joint probability density function of $Y_{(1)}$ and $Y_{(5)}$. [10]

c. Sketch the region $Y_{(5)} - Y_{(1)} > 0.8$ on the support set and then, set up the integrals to compute $P(Y_{(5)} - Y_{(1)} > 0.8)$. [10]