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NAME ______________________ DECEMBER 10, 2007

MATH 441 - MATHEMATICAL STATISTICS I LONG EXAM IV

Instructions: *Include all relevant work to get full credit. Write your solutions using proper notations.*

1. A brand of water softener salt comes in bags marked "net weight 40 lbs." The company claims that the bags contain an average of 40 pounds of salt and that the standard deviation of the weights is 3 pounds. Assume this to be true.

a. Let Y represent the net weight of a bag of water softener salt from this company. State the sampling distribution of $\bar{Y}$ for a sample of size 36. [3]

b. What is the justification for the sampling distribution of $\bar{Y}$ given in part (a)? [2]

c. What is the probability that the mean of 36 bags of this salt will weigh less than 39 lbs? [5]

2. A rental car company would like to know if its customers put more miles on sports cars than they do on luxury cars. A random sample of 13 sports cars registered an average of 55.4 miles per day with a standard deviation of 12.1 miles per day. An independent random sample of 15 luxury cars averaged 47.9 miles per day with a standard deviation of 9.5. Since the larger sample standard deviation is not more than twice the smaller, it would be reasonable to assume that $\sigma_1^2 = \sigma_2^2$.

a. Construct a 98% confidence interval for the difference in the true mean number of miles per day between sport rental cars and luxury rental cars. Interpret your interval. [12]

b. What must be assumed in order for the CI in part (a) to be valid? [2]

c. Construct a 95% confidence interval for the population standard deviation of the miles per day that customers put on sports cars. Interpret the interval. [10]

3. Determine the required <u>common</u> sample size for two independent samples in order to construct a 90% confidence interval for the <u>difference</u> in population proportions if the error of estimation is to be no more than 0.03? [10]

4. Let $Y_1, Y_2, \ldots, Y_{10}$ denote a random sample of size 10 from a normal distribution with mean 0 and variance 2.

a. Find the distribution of $\frac{10\bar{Y}^2}{\sigma^2}$. Justify your answer. [10]

b. Using the pivotal quantity $\frac{10\bar{Y}^2}{\sigma^2}$, derive the formulas for the lower and upper bounds of a two-sided $(1-\alpha)100\%$ confidence interval for σ^2. [10]

c. Show that the distribution of $\frac{10\bar{Y}^2}{S^2}$ is F(1, 9). [10]

d. Could the expression $\frac{10\bar{Y}^2}{S^2}$ from part (b) be used as a pivotal quantity for σ^2? Explain your answer. [3]

5. Suppose that $Y_1, Y_2, \ldots, Y_n$ denotes a random sample of size n from a population with an exponential distribution whose density is given by

$$f(y) = \frac{1}{\theta}e^{-y/\theta}, \quad y > 0.$$

Let $Y_{(1)} = min(Y_1, Y_2, \ldots, Y_n)$ denotes the smallest-order statistic.

a. Show that $\hat{\theta} = nY_{(1)}$ is an unbiased estimator of θ. [15]

b. Determine MSE$(\hat{\theta})$. [10]

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1. a) $\bar{Y} \approx N\left(\mu = 40, \ \sigma_{\bar{y}} = \frac{3}{\sqrt{36}}\right)$

b) Central Limit Theorem

c) $P(\bar{Y} < 39) = P\left(\frac{\bar{y} - 40}{1/2} < \frac{39-40}{1/2}\right) = P(z < -2) = .0228$

2. a) $s_p = \sqrt{\frac{(13-1)12.1^2 + (15-1)9.5^2}{13+15-2}} \approx \sqrt{\frac{3020.42}{26}} \approx \sqrt{116.17} \approx 10.78$

Let μ_s = mean miles on sports cars.
μ_L = " " " Luxury cars.

The 98% C.I. for $(\mu_s - \mu_L)$ is given by

$$(\bar{X}_s - \bar{X}_L) \pm t_{\frac{\alpha}{2}, 26} \cdot s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (55.4 - 47.9) \pm 2.479 \, (10.78) \sqrt{\frac{1}{13} + \frac{1}{15}}$$

$$= 7.5 \pm 10.13$$

$$= [-2.63, 17.63]$$

We are 98% confident that $(\mu_s - \mu_L)$ is between -2.63 and 17.63.

b) The two samples are independent and normally distributed.

c) The 95% C.I. for σ_s^2

$$\left(\frac{(13-1)(12.1^2)}{23.3367}, \frac{(13-1)(12.1^2)}{4.4038}\right) = [75.29, 398.96]$$

Hence, the 95% C.I. for σ_s

$$= [8.68, 19.97].$$

We are 95% confident that σ_s is between 8.68 and 19.97.

3. $$M.E. = Z_{\frac{\alpha}{2}} \sqrt{\frac{p_1(1-p_1)}{n} + \frac{p_2(1-p_2)}{n}}$$

$$\Rightarrow \sqrt{n} = \frac{Z_{\alpha/2}}{ME.} \sqrt{p_1(1-p_1)+p_2(1-p_2)}$$

$$\Rightarrow n \leq \frac{z^2}{(M.E.)^2}\, 2\left(\frac{1}{4}\right) = \frac{(1.645)^2}{(.03)^2}\left(\frac{1}{2}\right) \approx 1503.35$$

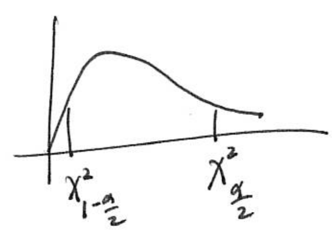
hence, use $n = 1504$

4. a) Since $Y_i \sim N(\mu=0, \sigma^2)$

$$\Rightarrow \bar{Y} \sim N\left(\mu=0,\ \sigma^2_{\bar{y}} = \frac{\sigma^2}{10}\right)$$

$$\Rightarrow \frac{\bar{y}-0}{\sigma/\sqrt{10}} \sim N(0,1)$$

$$\Rightarrow 10\,\frac{\bar{y}^2}{\sigma^2} \sim \chi^2_1$$



b) $$P\left(\chi^2_{1-\frac{\alpha}{2}} < \frac{10\bar{Y}^2}{\sigma^2} < \chi^2_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{1}{\chi^2_{\alpha/2}} < \frac{\sigma^2}{10\bar{Y}^2} < \frac{1}{\chi^2_{1-\frac{\alpha}{2}}}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{10\bar{Y}^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{10\bar{Y}^2}{\chi^2_{1-\frac{\alpha}{2}}}\right) = 1-\alpha$$

Hence, the $(1-\alpha)100\%$ C.I. for σ^2 is

$$\left[\frac{10\bar{Y}^2}{\chi^2_{\alpha/2}},\ \frac{10\bar{Y}^2}{\chi^2_{1-\alpha/2}}\right]$$

c) ~~When~~ We know that $\frac{10\bar{Y}^2}{\sigma^2} \sim \chi^2_1$ and $\frac{(10-1)S^2}{\sigma^2} \sim \chi^2_9$

Therefore, $$\frac{\frac{10\bar{Y}^2}{\sigma^2}/1}{\frac{(9)S^2}{\sigma^2}/9} = \frac{10\bar{Y}^2}{S^2} \sim F_{(1,9)}$$

d) No, because it does not contain any parameter.

5. $Y_i \sim f(y) = \frac{1}{\theta} e^{-y/\theta}, \quad y > 0 \quad \Rightarrow$ since $Y \sim exp(\theta)$ $E(Y) = \theta$ and $V(Y) = \theta^2$

$\Rightarrow F_Y(y) = 1 - e^{-y/\theta}, \quad y > 0$

a) Hence, $F_{Y_{(1)}}(y) = 1 - [1 - F_Y(y)]^n = 1 - (e^{-y/\theta})^n = 1 - e^{-y/(\frac{\theta}{n})}, \quad y > 0$

This means that $Y_{(1)} \sim exp(\frac{\theta}{n})$

Therefore, $E(Y_{(1)}) = \frac{\theta}{n}$ and so $E(\hat{\theta}) = E(n \cdot Y_{(1)}) = n(\frac{\theta}{n}) = \theta$. ■

b) $MSE(\hat{\theta}) = Var(\hat{\theta})$ since $\hat{\theta}$ is unbiased.

$= V(n \cdot Y_{(1)})$

$= n^2 V(Y_{(1)}) = n^2 \left[\frac{\theta^2}{n^2} \right] = \theta^2$.