Two-Factor Studies - One Case per Treatment

- **Factors Do Not Interact**

ANOVA Table

Source	DF	Sum of Squares	Mean Square	F
Factor A	$a-1$	$SSA = b\sum_{i=1}^{a}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2$	$MSA = \frac{SSA}{a-1}$	$F_{\mathrm{A}} = \frac{MSA}{MSAB}$
Factor B	$b-1$	$SSB = a\sum_{j=1}^{b}(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2$	$MSB = \frac{SSB}{b-1}$	$F_{\mathrm{B}} = \frac{MSB}{MSAB}$
Error	$(a-1)(b-1)$	$SSAB$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	
Total	$ab-1$	$SSTO = \sum_{i=1}^{a}\sum_{j=1}^{b}(Y_{ij} - \bar{Y}_{\cdot\cdot})^2$		

Note:

1. $SSAB = \sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{Y}_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y}_{\cdot\cdot})^2$

2. $E(MSA) = \sigma^2 + b\frac{\sum(\mu_{i\cdot} - \mu_{\cdot\cdot})^2}{a-1}$.

3. $E(MSB) = \sigma^2 + a\frac{\sum(\mu_{\cdot j} - \mu_{\cdot\cdot})^2}{b-1}$.

4. $E(MSAB) = \sigma^2$.

- **Example:** An analyst in an insurance commissioner's office studied the premiums for automobile insurance charged by an insurance company in six cities. The six cities were selected to represent different regions of the state and different sizes of cities. The table below shows the amounts of three-month premiums charged by the automobile insurance firm for a specific type and amount of coverage in a given risk category for each of the six cities, classified by size for city (factor A) and geographic region (factor B).

	East	West
Small	140	100
Medium	210	180
Large	220	200

Sample R commands:

```
resp=c(140,100,210,180,220,200)
size=c(1,1,2,2,3,3); region=c(1,2,1,2,1,2)
A=factor(size); B= factor(region)
result=aov(resp~A+B)
anova(result)
```

- **Estimation of Treatment Mean**

  Occasionally when no-interaction model is employed with one case per treatment, there is interest in estimating a treatment mean μ_{ij}. A good way to make use of the model assumption of no interaction, is to estimate μ_{ij} by

  $$\hat{\mu}_{ij} = \bar{Y}_{i\cdot} + \bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot}$$

  This estimator is an improved estimator compared to $\bar{Y}_{ijk} = Y_{ij}$ because it has minimum variance in the class of unbiased linear estimators.

- **Tukey Test for Additivity**

  This test is useful in testing whether or not the two factors in a two-factor study interact when $n = 1$. Suppose we consider that $(\alpha\beta)_{ij} = D\alpha\beta$, where D is some constant. Note that when $D = 0$, all $(\alpha\beta)_{ij}$ will be 0 and hence, we have the additive model and no interaction effect.

  1. $H_o : D = 0$ vs. $H_a : D \neq 0$

  2. *Identify appropriate test statistic*:

  $F^* = \dfrac{(SSAB^*/1)}{(SSRem^*/(ab - a - b))} \sim F(df_1 = 1, df_2 = ab - a - b)$

  where,

  a. $\hat{D} = \dfrac{\sum_i \sum_j (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})Y_{ij}}{\sum_i (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 \sum_j (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2}$

  b. $SSAB^* = \sum_i \sum_j \hat{D}^2 (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2 = \dfrac{[\sum_i \sum_j (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})Y_{ij}]^2}{\sum_i (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2 \sum_j (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2}$

  c. $SSTO = SSA + SAB + SSAB^* + SSRem^*$

  d. $SSRem^* = SSTO - SSA - SSB - SSAB^* = SSAB - SSAB^*$ is the remainder sum of squares:

  3. *Specify α and the rejection rule*:

  4. *Calculate the observed F^* and it's corresponding p-value*:

  5. *Write you conclusion*:

- **Insurance Premium Example**

	East	West
Small	140	100
Medium	210	180
Large	220	200