Multi-Factor Studies

When three or more factors are studied simultaneously, the model and analysis employed are straightforward extensions of the two-factor case.

- ANOVA Model for Three-Factor Studies

$$Y_{ijkm} = \mu_{\cdots} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkm}$$

ANOVA Table

Source	DF	Sum of Squares	Mean Square	F
Factor A	$a-1$	$SSA = \sum_{i=1}^{a} nbc(\bar{Y}_{i\cdots} - \bar{Y}_{\cdots\cdot})^2$	$MSA = \frac{SSA}{a-1}$	$F_{\text{A}} = \frac{MSA}{MSE}$
Factor B	$b-1$	$SSB = \sum_{j=1}^{b} nac(\bar{Y}_{\cdot j\cdot\cdot} - \bar{Y}_{\cdots\cdot})^2$	$MSB = \frac{SSB}{b-1}$	$F_{\text{B}} = \frac{MSB}{MSE}$
Factor C	$c-1$	$SSC = \sum_{k=1}^{c} nab(\bar{Y}_{\cdot\cdot k\cdot} - \bar{Y}_{\cdots\cdot})^2$	$MSC = \frac{SSC}{c-1}$	$F_{\text{C}} = \frac{MSC}{MSE}$
AB Interaction	$(a-1)(b-1)$	$SSAB$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	$F_{\text{AB}} = \frac{MSAB}{MSE}$
AC Interaction	$(a-1)(c-1)$	$SSAC$	$MSAC = \frac{SSAC}{(a-1)(c-1)}$	$F_{\text{AC}} = \frac{MSAC}{MSE}$
BC Interaction	$(b-1)(c-1)$	$SSBC$	$MSBC = \frac{SSBC}{(b-1)(c-1)}$	$F_{\text{BC}} = \frac{MSBC}{MSE}$
ABC Interaction	$(a-1)(b-1)(c-1)$	$SSABC$	$MSABC = \frac{SSABC}{(a-1)(b-1)(c-1)}$	$F_{\text{ABC}} = \frac{MSABC}{MSE}$
Error	$abc(n-1)$	$SSE = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{m=1}^{n}(Y_{ijkm} - \bar{Y}_{ijk\cdot})^2$	$MSE = \frac{SSE}{abc(n-1)}$	
Total	$nabc-1$	$SSTO = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{m=1}^{n}(Y_{ijkm} - \bar{Y}_{\cdots\cdot})^2$		

Note:

1. $SSAB = nc\sum_i\sum_j(\bar{Y}_{ij\cdot\cdot} - \bar{Y}_{i\cdots} - \bar{Y}_{\cdot j\cdot\cdot} + \bar{Y}_{\cdots\cdot})^2$
2. $SSAC = nb\sum_i\sum_k(\bar{Y}_{i\cdot k\cdot} - \bar{Y}_{i\cdots} - \bar{Y}_{\cdot\cdot k\cdot} + \bar{Y}_{\cdots\cdot})^2$
3. $SSBC = na\sum_j\sum_k(\bar{Y}_{\cdot jk\cdot} - \bar{Y}_{\cdot j\cdot\cdot} - \bar{Y}_{\cdot\cdot k\cdot} + \bar{Y}_{\cdots\cdot})^2$
4. $SSABC = n\sum_i\sum_j\sum_k(\bar{Y}_{ijk\cdot} - \bar{Y}_{ij\cdot\cdot} - \bar{Y}_{i\cdot k\cdot} - \bar{Y}_{\cdot jk\cdot} + \bar{Y}_{i\cdots} + \bar{Y}_{\cdot j\cdot\cdot} + \bar{Y}_{\cdot\cdot k\cdot} - \bar{Y}_{\cdots\cdot})^2$
5. $SSTR = SSA + SSB + SSC + SSAB + SSAC + SSBC + SSABC = n\sum_i\sum_j\sum_k(\bar{Y}_{ijk\cdot} - \bar{Y}_{\cdots\cdot})^2$.
6. $SSTO = SSTR + SSE$.
7. $E(MSA) = \sigma^2 + bcn\frac{\sum(\mu_{i\cdot\cdot} - \mu_{\cdots})^2}{a-1} = \sigma^2 + bcn\frac{\sum\alpha_i^2}{a-1}$.
8. $E(MSB) = \sigma^2 + acn\frac{\sum(\mu_{\cdot j\cdot} - \mu_{\cdots})^2}{b-1} = \sigma^2 + acn\frac{\sum\beta_j^2}{b-1}$.
9. $E(MSC) = \sigma^2 + abn\frac{\sum(\mu_{\cdot\cdot k} - \mu_{\cdots})^2}{c-1} = \sigma^2 + abn\frac{\sum\gamma_k^2}{c-1}$.
10. $E(MSAB) = \sigma^2 + cn\frac{\sum\sum(\mu_{ij\cdot} - \mu_{i\cdot\cdot} - \mu_{\cdot j\cdot} + \mu_{\cdots})^2}{(a-1)(b-1)} = \sigma^2 + cn\frac{\sum(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$.
11. $E(MSAC) = \sigma^2 + bn\frac{\sum\sum(\mu_{i\cdot k} - \mu_{i\cdot\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdots})^2}{(a-1)(c-1)} = \sigma^2 + bn\frac{\sum(\alpha\gamma)_{ik}^2}{(a-1)(c-1)}$.
12. $E(MSBC) = \sigma^2 + an\frac{\sum\sum(\mu_{\cdot jk} - \mu_{\cdot j\cdot} - \mu_{\cdot\cdot k} + \mu_{\cdots})^2}{(b-1)(c-1)} = \sigma^2 + an\frac{\sum(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$.
13. $E(MSABC) = \sigma^2 + n\frac{\sum\sum\sum(\mu_{ijk} - \mu_{ij\cdot} - \mu_{i\cdot k} - \mu_{\cdot jk} + \mu_{i\cdot\cdot} + \mu_{\cdot j\cdot} + \mu_{\cdot\cdot k} - \mu_{\cdots})^2}{(a-1)(b-1)(c-1)} = \sigma^2 + n\frac{\sum(\alpha\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$.
14. $E(MSE) = \sigma^2$.

- **Stress Test Example**

  The effects of gender of subject (factor A: male or female), body fat of subject (measured in percent, factor B: low or high), and smoking history of subject(factor C: light or heavy) on exercise tolerance (Y) were studied in a small-scale investigation of persons 25 to 35 years old. Exercise tolerance was measured in minutes until fatigue occurs while the subject is performing on a bicycle apparatus. Three subjects for each gender-body fat-smoking history group were given the exercise tolerance stress test. The results are shown in the table below:

	Light	Heavy
Low fat		
Male	24.1	17.6
	29.2	18.8
	24.6	23.2
Female	20.0	14.8
	21.9	10.3
	17.6	11.3
High fat		
Male	14.6	14.9
	15.3	20.4
	12.3	12.8
Female	16.1	10.1
	9.3	14.4
	10.8	6.1

- **R commands:**

```
data=read.csv("StressTest.csv",header=T)
attach(data)

A=factor(gender)
B=factor(fat)
C=factor(smoking)
result=aov(tolerance~A*B*C)
anova(result)
Response: tolerance
          Df  Sum Sq Mean Sq F value    Pr(>F)
A          1 176.584 176.584 18.9155 0.0004971 ***
B          1 242.570 242.570 25.9839 0.0001076 ***
C          1  70.384  70.384  7.5394 0.0143574 *
A:B        1  13.650  13.650  1.4622 0.2441432
A:C        1  11.070  11.070  1.1859 0.2922989
B:C        1  72.454  72.454  7.7612 0.0132205 *
A:B:C      1   1.870   1.870  0.2004 0.6604336
Residuals 16 149.367   9.335
```

- **Kimball Inequality.** The Kimball inequality for the family level of significance α in a three-factor study when the family consists of the combined set of seven tests, including three on main effects, three on two-factor interactions, and one on three-factor interactions, is:

$$\alpha < 1 - (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_7)$$

  where α_i is the level of significance for the ith test.

- **Strategy for Analysis**

  1. Examine whether or not important three-factor interactions exist.

  2. If no important three-factor interactions exist, determine whether or not important two-factor interactions are present.

  3. If no important two-factor or higher-order interactions are present, examine the main effects. For important A, B, and C main effects, describe the nature of these effects in terms of the factor level means $\mu_{i\cdot\cdot}$, $\mu_{\cdot j\cdot}$, and $\mu_{\cdot\cdot k}$, respectively.

  4. If three-factor interactions are important, consider whether they can be made unimportant by meaningful simple transformation of scale. If so, make the transformation and proceed as in step 2.

  5. For important three-factor interactions that cannot be made unimportant by a simple transformation, which is often the case, analyze the three factors jointly in terms of the treatment means μ_{ijk}.

  6. If there is just one important two-factor interaction, analyze the effects jointly in terms of the appropriate two-factor treatment means $\mu_{ij\cdot}$, $\mu_{i\cdot k}$, and$\mu_{\cdot jk}$

  7. If there are two or three important two-factor interactions in a three-factor study, analyze the three factors jointly in terms of the treatment means μ_{ijk}.

- **Meaning of ANOVA Model Elements.** Use the example in the tables below to find $\mu_{\cdot 11}$, $\mu_{\cdot 1\cdot}$,$\mu_{\cdot\cdot 1}$, $\mu_{1\cdot\cdot}$, $\mu_{\cdot\cdot\cdot}$, α_1, α_2, γ_1, and γ_2.

High IQ			
	Young	Middle	Old
Male	$\mu_{111} = 9$	$\mu_{121} = 12$	$\mu_{131} = 18$
Female	$\mu_{211} = 9$	$\mu_{221} = 10$	$\mu_{231} = 14$

Low IQ			
	Young	Middle	Old
Male	$\mu_{112} = 19$	$\mu_{122} = 20$	$\mu_{132} = 21$
Female	$\mu_{212} = 19$	$\mu_{222} = 20$	$\mu_{232} = 21$