Random and Mixed Effects Models

- **Types of ANOVA Models**

  1. ANOVA Model I - All factor levels are fixed (fixed factors).

  2. ANOVA Model II - All factors are random.

  3. ANOVA Model III - Some factors are fixed and some are random.

- **ANOVA Model II with one factor**. Random Cell Means Model (Random ANOVA Model)

$$Y_{ij} = \mu_i + \epsilon_{ij}$$

  where:

  1. μ_i are independent $N(\mu., \sigma_\mu^2)$

  2. ϵ_{ij} are independent $N(0, \sigma^2)$

  3. μ_i and ϵ_{ij} are independent random variables

  4. $i = 1, \ldots, r$; $j = 1, \ldots, n$ (Equal sample sizes)

- **Apex Enterprises Example**. Five personnel officers were selected at random, and four prospective employee candidates were assigned at random to each selected officer. The results are shown in the table below. Determine if the mean ratings of the personnel officers differ.

Officer	Candidate 1	Candidate 2	Candidate 3	Candidate 4
A	76	65	85	74
B	59	75	81	67
C	49	63	61	46
D	74	71	85	89
E	66	84	80	79

- **R commands:**

```
ratings=c(76,65,85,74,59,75,81,67,49,63,61,46,74,71,85,89,66,84,80,79)
officer=gl(5,4)

result=aov(ratings~officer)
anova(result)
Response: ratings
          Df Sum Sq Mean Sq F value   Pr(>F)
officer    4 1579.7  394.93   5.389 0.006803 **
Residuals 15 1099.2   73.28

mean(ratings)
[1] 71.45
```

- **Estimation of** $\mu.$: Note that $E\{Y_{ij}\} = \mu.$ and $\sigma^2\{\bar{Y}_{ij}\} = \sigma_Y^2 = \sigma_\mu^2 + \sigma^2$.

  1. $\hat{\mu}. = \bar{Y}..$

  2. $\sigma^2\{\bar{Y}..\} = \frac{\sigma_\mu^2}{r} + \frac{\sigma^2}{rn} = \frac{n\sigma_\mu^2 + \sigma^2}{rn} = \frac{E\{MSTR\}}{rn}$

  3. $s^2\{\bar{Y}..\} = \frac{MSTR}{rn}$

  4. The $(1-\alpha)100\%$ confidence interval: $\bar{Y}.. \pm t_{[1-\alpha/2; r-1]} s\{\bar{Y}..\}$

5. *Example.* Management of Apex Enterprises wishes to estimate the mean rating for all prospective employees by all personnel officers with 90% confidence interval.

- **Estimation of $\sigma_\mu^2/(\sigma_\mu^2+\sigma^2)$**: This quantity reveals the effect of the extent of variation between the μ_i. Note that

$$\frac{MSTR}{n\sigma_\mu^2+\sigma^2} \div \frac{MSE}{\sigma} \sim F[r-1, r(n-1)]$$

1. The $(1-\alpha)100\%$ confidence interval for σ_μ^2/σ^2 is $[L, U]$, where:

$$L = \frac{1}{n}\left[\frac{MSTR}{MSE}\left(\frac{1}{F[1-\alpha/2; r-1, r(n-1)]}\right) - 1\right]$$

and

$$U = \frac{1}{n}\left[\frac{MSTR}{MSE}\left(\frac{1}{F[\alpha/2; r-1, r(n-1)]}\right) - 1\right]$$

2. The $(1-\alpha)100\%$ confidence interval for $\sigma_\mu^2/(\sigma_\mu^2+\sigma^2)$ is $[L^*, U^*]$, where:

$$L^* = \frac{L}{1+L} \quad \text{and} \quad U^* = \frac{U}{1+U}$$

3. *Example.* Obtain a 90% confidence interval for $\sigma_\mu^2/(\sigma_\mu^2+\sigma^2)$ for the Apex Enterprises example.

- **Estimation of σ^2**: Note that $E(MSE) = \sigma^2$ and

$$\frac{r(n-1)MSE}{\sigma^2} \sim \chi^2[r(n-1)]$$

1. The $(1-\alpha)100\%$ confidence interval for σ^2 is

$$\frac{r(n-1)MSE}{\chi^2[1-\alpha/2; r(n-1)]} \quad \text{and} \quad \frac{r(n-1)MSE}{\chi^2[\alpha/2; r(n-1)]}$$

2. *Example.* Construct a 90% confidence interval for σ^2 for the Apex Enterprises example.

- **Estimation of σ_μ^2**: Note that $E(MSE) = \sigma^2$ and $E(MSTR) = \sigma^2 + n\sigma_\mu^2$. Hence,

$$\sigma_\mu^2 = \frac{E(MSTR) - E(MSE)}{n}$$

  1. An unbiased estimator of σ_μ^2 is $s_\mu^2 = \dfrac{MSTR - MSE}{n}$.

  2. *Satterthwaite Procedure.* An approximate $(1-\alpha)100\%$ confidence interval for σ_μ^2 is

$$\frac{(df)s_\mu^2}{\chi^2[1-\alpha/2; df]} \quad \text{and} \quad \frac{(df)s_\mu^2}{\chi^2[\alpha/2; df]}$$

  where,

$$df = \frac{(ns_\mu^2)^2}{\frac{(MSTR)^2}{r-1} + \frac{(MSE)^2}{r(n-1)}}$$

  3. *Example.* Using the Satterthwaite procedure, obtain a 90% confidence interval for σ_μ^2 for the Apex Enterprises example.

- **Random Factor Effects Model**. Let $\tau_i = \mu_i - \mu.$, then the random factor effects model is expressed as follows:

$$Y_{ij} = \mu. + \tau_i + \epsilon_{ij}$$

  where:

  1. $\mu.$ is a constant component common to all observations

  2. τ_i are independent $N(0, \sigma_\mu^2)$

  3. ϵ_{ij} are independent $N(0, \sigma^2)$

  4. τ_i and ϵ_{ij} are independent

  5. $i = 1, \ldots, r$; $j = 1, \ldots, n$ (Equal sample sizes)