Random and Mixed Effects Models

- **Two-Factor Studies - ANOVA Model II**. Both factors A and B are random.

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ij}$$

where:

1. $\mu_{..}$ is a constant component common to all observations

2. $\alpha_i, \beta_j, (\alpha\beta)_{ij}$ are independent normal random variables with zero means and respective variances σ^2_α, σ^2_β, $\sigma^2_{\alpha\beta}$

3. ϵ_{ij} are independent $N(0, \sigma^2)$

4. $\alpha_i, \beta_j, (\alpha\beta)_{ij}$ and ϵ_{ij} are pairwise independent

5. $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$ (Equal sample sizes)

- **Two-Factor Studies - ANOVA Model III**. Factor A is fixed and B is random.

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ij}$$

where:

1. $\mu_{..}$ is a constant component common to all observations

2. α_i are constants subject to the restriction $\sum_i \alpha_i = 0$

3. β_j are independent $N(0, \sigma^2_\beta)$

4. $(\alpha\beta)_{ij}$ are $N\left(0, \left(\frac{a-1}{a}\right)\sigma^2_{\alpha\beta}\right)$, subject to the restrictions:

  a. $\sum_i (\alpha\beta)_{ij} = 0$ for all j

  b. $\sigma\{(\alpha\beta)_{ij}, (\alpha\beta)_{i'j'}\} = -\frac{1}{a}\sigma^2_{\alpha\beta}$

5. ϵ_{ij} are independent $N(0, \sigma^2)$

6. $\beta_j, (\alpha\beta)_{ij}$ and ϵ_{ij} are pairwise independent

7. $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$ (Equal sample sizes)

- **Expected Mean Squares**

Mean Square	df	Fixed ANOVA Model (A and B fixed)	Random ANOVA Model (A and B random)	Mixed ANOVA Model (A fixed, B random)
MSA	$a-1$	$\sigma^2 + nb\frac{\sum_i \alpha_i^2}{a-1}$	$\sigma^2 + nb\sigma^2_\alpha + n\sigma^2_{\alpha\beta}$	$\sigma^2 + nb\frac{\sum_i \alpha_i^2}{a-1} + n\sigma^2_{\alpha\beta}$
MSB	$b-1$	$\sigma^2 + na\frac{\sum_j \beta_j^2}{b-1}$	$\sigma^2 + na\sigma^2_\beta + n\sigma^2_{\alpha\beta}$	$\sigma^2 + na\sigma^2_\beta$
$MSAB$	$(a-1)(b-1)$	$\sigma^2 + n\frac{\sum_i \sum_j (\alpha_i\beta_j)^2_{ij}}{(a-1)(b-1)}$	$\sigma^2 + n\sigma^2_{\alpha\beta}$	$\sigma^2 + n\sigma^2_{\alpha\beta}$
MSE	$(n-1)ab$	σ^2	σ^2	σ^2

- **Test Statistics for Balanced Two-Factor ANOVA Models**

Test for Presence of Effects of:	Fixed ANOVA Model (A and B fixed)	Random ANOVA Model (A and B random)	Mixed ANOVA Model (A fixed, B random)
Factor A	MSA/MSE	$MSA/MSAB$	$MSA/MSAB$
Factor B	MSB/MSE	$MSB/MSAB$	MSB/MSE
AB Interactions	$MSAB/MSE$	$MSAB/MSE$	$MSAB/MSE$

- **ANOVA Model II Example**. Consider an investigation of the effects of machine operators (factor A) and machines (factor B) on the number of pieces produced in a day. Five operators and three machines are used in the study. Yet the inferences are not to be confined to the particular five operators and three machines participating in the study, but rather they are to pertain to all operators and all machines available to the company.

- **ANOVA Model III Example**. Consider an investigation of the effects four different training methods (factor A, fixed) and five instructors (factor B, random) upon learning in a company training program. Four classes were assigned to each training method-instruction combination. The response variable of interest was the mean improvement per student in the class at the end of the training program. Parts of the ANOVA result are given below:

ANOVA Table for Mixed Model

Source of Variation	DF	Sum of Squares	Mean Square	F
Factor A		42.1		
Factor B		53.9		
AB Interactions		46.7		
Error		126.4		
Total		269.1		

- **Estimation of Variance Components**. When a random factor has significant main effects, we often wish to estimate the magnitude of the variance component.

  1. Note $\sigma_\beta^2 = \dfrac{E(MSB) - E(MSE)}{na}$

  2. An unbiased estimator of σ_β^2 is $s_\beta^2 = \dfrac{MSB - MSE}{na}$

  3. *Satterthwaite Procedure.* An approximate $(1-\alpha)100\%$ confidence interval for σ_β^2 is

$$\frac{(df)s_\beta^2}{\chi^2[1-\alpha/2; df]} \quad \text{and} \quad \frac{(df)s_\beta^2}{\chi^2[\alpha/2; df]}$$

  where,

$$df = \frac{(nas_\beta^2)^2}{\frac{(MSB)^2}{b-1} + \frac{(MSE)^2}{(n-1)ab}}$$