Random and Mixed Effects Models

- **ANOVA Model III Example**. Consider an investigation of the effects four different training methods (factor A, fixed) and five instructors (factor B, random) upon learning in a company training program. Four classes were assigned to each training method-instruction combination. The response variable of interest was the mean improvement per student in the class at the end of the training program. Parts of the ANOVA result are given below:

ANOVA Table for Mixed Model

Source of Variation	DF	Sum of Squares	Mean Square	F	$p-$value
Factor A	3	42.1	14.03	3.61	0.0458
Factor B	4	53.9	13.475	6.39	0.00024
AB Interactions	12	46.7	3.89	1.84	0.06
Error	60	126.4	2.11		
Total	79	269.1			

- **Estimation of Fixed Effects in Balanced Mixed Model**. When a fixed factor has significant main effects, we often wish to estimate these effects.

  1. An unbiased estimator of the fixed effect α_i is $\hat{\alpha}_i = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}$
  2. An unbiased estimator of the marginal mean $\mu_{i\cdot}$ is $\hat{\mu}_{i\cdot} = \bar{Y}_{i\cdot\cdot}$
  3. An unbiased estimator of the contrast $L = \sum c_i\alpha_i$ is $\hat{L} = \sum c_i\hat{\alpha}_i = \sum c_i(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}) = \sum c_i\bar{Y}_{i\cdot\cdot}$
  4. An unbiased estimator of $\sigma^2\{\hat{\alpha}_i\} = \frac{\sigma^2+n\sigma^2_{\alpha\beta}}{bn} = \frac{E(MSAB)}{bn}$ is $s^2\{\hat{\alpha}_i\} = \frac{MSAB}{bn}$
  5. An unbiased estimator of $\sigma^2\{\hat{L}\} = \sum c_i^2\sigma^2\{\hat{\alpha}_i\} = \frac{E(MSAB)}{bn}\sum c_i^2$ is $s^2\{\hat{L}\} = \frac{MSAB}{bn}\sum c_i^2$
  6. The test statistic $\dfrac{\hat{L}-L}{s\{\hat{L}\}}$ is distributed as $t_{[(a-1)(b-1)]}$
  7. The $(1-\alpha)$ confidence limits for L are: $\hat{L} \pm t_{[1-\alpha/2;(a-1)(b-1)]} * s\{\hat{L}\}$

- In the previous example, construct a 95% confidence interval for the difference $L = \alpha_1 - \alpha_2 = \mu_{1\cdot} - \mu_{2\cdot}$ in the mean improvements between training methods 1 and 2. The relevant sample results are: $\bar{Y}_{1\cdot\cdot} = 43.1$ and $\bar{Y}_{2\cdot\cdot} = 40.8$.

- **Multiple Comparison Procedures**. Multiple comparison procedures that we studied before are applicable for the main effects of the fixed factor.

  1. Tukey: $T = \frac{1}{\sqrt{2}}q[1-\alpha; a, (a-1)(b-1)]$
  2. Scheffe: $S^2 = (a-1)f[1-\alpha; (a-1), (a-1)(b-1)]$
  3. Bonferroni: $B = t[1-\alpha/(2*g), (a-1)(b-1)]$

- **Confidence Intervals for Marginal Means**. Use the *Satterthwaite Procedure* to get an approximate confidence interval. Note that

$$\sigma^2\{\hat{\mu}_{i\cdot}\} = c_1 E(MSAB) + c_2 E(MSB)$$

where,

$$c_1 = \frac{a-1}{nab} \quad \text{and} \quad c_2 = \frac{1}{nab}$$

Hence, an unbiased estimator of $\sigma^2\{\hat{\mu}_{i\cdot}\}$ is $s^2\{\hat{\mu}_{i\cdot}\} = c_1 MSAB + c_2 MSB$. The degrees of freedom associated with this estimator is

$$df = \frac{(s^2\{\hat{\mu}_{i\cdot}\})^2}{\frac{(c_1 MSAB)^2}{(a-1)(b-1)} + \frac{(c_2 MSB)^2}{b-1}}$$

The approximate $1-\alpha$ confidence limits for $\mu_{i\cdot}$ are: $\hat{\mu}_{i\cdot} \pm t_{[1-\alpha/2;df]} * s\{\hat{\mu}_{i\cdot}\}$

- **Randomized Complete Block Design: Random Block Effects.** Additive Model

$$Y_{ij} = \mu_{\cdot\cdot} + \rho_i + \tau_i + \epsilon_{ij}$$

where:

1. $\mu_{\cdot\cdot}$ is a constant

2. ρ_i are independent $N(0, \sigma^2\rho)$

3. τ_i are constants subject to the restriction $\sum \tau_i = 0$

4. ϵ_{ij} are independent $N(0, \sigma^2)$, and independent of the ρ_i

5. $i = 1, \ldots, n_b; j = 1, \ldots, r$

6. Note $E(Y_{ij}) = \mu_{\cdot\cdot} + \tau_j$ and $\sigma^2\{Y_{ij}\} = \sigma_Y^2 = \sigma_\rho^2 + \sigma^2$

- **Randomized Complete Block Design: Random Block Effects.** Interaction Model

$$Y_{ij} = \mu_{\cdot\cdot} + \rho_i + \tau_i + (\rho\tau)_{ij} + \epsilon_{ij}$$

where:

1. $(\rho\tau)_{ij}$ are $N\left(0, \frac{r-1}{r}\sigma_{\rho\tau}^2\right)$ subject to $\sum_j (\rho\tau)_{ij} = 0$ for all j

2. Note $E(Y_{ij}) = \mu_{\cdot\cdot} + \tau_j$ and $\sigma^2\{Y_{ij}\} = \sigma_Y^2 = \sigma_\rho^2 + \frac{r-1}{r}\sigma_{\rho\tau}^2 + \sigma^2$

1. A researcher investigated the improvement in learning in third-grade classes by augmenting the teacher with one or two teaching assistants. Ten schools were selected at random and three third-grade classes in each school were utilized in the study. In each school, one class was randomly chosen to have no teaching assistant, one class was randomly chosen to have one teaching assistant, and the third class was assigned two teaching assistants. The amount of learning by the class at the end of the school year, suitably measure, was the response variable.

2. In a study of the effectiveness of four different dosages of a drug, 20 litters of mice, each consisting of four mice, were utilized. The 20 litters (blocks) here may be viewed as a random sample from the population of all litters that could have been used for the study.

- **ANOVA Table for Randomized Complete Block Design.** Block Effects Random, Treatment Effects Fixed.

Source	DF	Sum of Squares	Mean Square	Additive	Interaction
Blocks	$n_b - 1$	$SSBL$	$MSBL$	$\sigma^2 + r\sigma_\rho^2$	$\sigma^2 + r\sigma_\rho^2$
Treatments	$r-1$	$SSTR$	$MSTR$	$\sigma^2 + n_b\frac{\sum\tau_j^2}{r-1}$	$\sigma^2 + \sigma_{\rho\tau}^2 + n_b\frac{\sum\tau_j^2}{r-1}$
Error	$(n_b-1)(r-1)$	SSE	MSE	σ^2	$\sigma^2 + \sigma_{\rho\tau}^2$
Total	$n_b r - 1$	$SSTO$			