Random and Mixed Effects Models

- **Three-Factor Studies.** ANOVA Model II - Random Factor Effects

$$Y_{ijkm} = \mu_{...} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkm}$$

where:

1. $\mu_{...}$ is a constant

2. $\alpha_i, \beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}, \epsilon_{ijkm}$ are independent normal random variables with expectations zero and respective variances $\sigma^2_\alpha, \sigma^2_\beta, \sigma^2_\gamma, \sigma^2_{\alpha\beta}, \sigma^2_{\alpha\gamma}, \sigma^2_{\beta\gamma}, \sigma^2_{\alpha\beta\gamma}, \sigma^2$

3. $i = 1, \ldots, a; j = 1, \ldots, b; k = 1, \ldots, c; m = 1, \ldots, n$ (Equal sample sizes)

4. Note $E(Y_{ijkm}) = \mu_{...}$ and $\sigma^2\{Y_{ijkm}\} = \sigma^2_Y = \sigma^2_\alpha + \sigma^2_\beta + \sigma^2_\gamma + \sigma^2_{\alpha\beta} + \sigma^2_{\alpha\gamma} + \sigma^2_{\beta\gamma} + \sigma^2_{\alpha\beta\gamma} + \sigma^2$

- **Expected Mean Squares.** ANOVA Model II - Random Factor Effects

Mean Square	df	Expected Mean Square
MSA	$a-1$	$\sigma^2 + nbc\sigma^2_\alpha + nc\sigma^2_{\alpha\beta} + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$
MSB	$b-1$	$\sigma^2 + nac\sigma^2_\beta + nc\sigma^2_{\alpha\beta} + na\sigma^2_{\beta\gamma} + n\sigma^2_{\alpha\beta\gamma}$
MSC	$c-1$	$\sigma^2 + nab\sigma^2_\gamma + nb\sigma^2_{\alpha\gamma} + na\sigma^2_{\beta\gamma} + n\sigma^2_{\alpha\beta\gamma}$
$MSAB$	$(a-1)(b-1)$	$\sigma^2 + nc\sigma^2_{\alpha\beta} + n\sigma^2_{\alpha\beta\gamma}$
$MSAC$	$(a-1)(c-1)$	$\sigma^2 + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$
$MSBC$	$(b-1)(c-1)$	$\sigma^2 + na\sigma^2_{\beta\gamma} + n\sigma^2_{\alpha\beta\gamma}$
$MSABC$	$(a-1)(b-1)(c-1)$	$\sigma^2 + n\sigma^2_{\alpha\beta\gamma}$
MSE	$(n-1)abc$	σ^2

- **Three-Factor Studies.** ANOVA Model III - Mixed Factor Effects (Fixed A, Random B and C)

$$Y_{ijkm} = \mu_{...} + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkm}$$

where:

1. $\mu_{...}$ is a constant

2. α_i are constants subject to $\sum_i \alpha_i = \sum_i (\alpha\beta)_{ij} = \sum_i (\alpha\gamma)_{ik} = \sum_i (\alpha\beta\gamma)_{ijk} = 0$

3. $\beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}$ are pairwise independent normal random variables with expectations zero and constant variances

4. ϵ_{ijkm} are independent $N(0, \sigma^2)$, and are independent of the other random components

5. $i = 1, \ldots, a; j = 1, \ldots, b; k = 1, \ldots, c; m = 1, \ldots, n$ (Equal sample sizes)

6. Note $E(Y_{ijkm}) = \mu_{...} + \alpha_i$ and $\sigma^2\{Y_{ijkm}\} = \sigma^2_Y = \sigma^2_\beta + \sigma^2_\gamma + \sigma^2_{\alpha\beta} + \sigma^2_{\alpha\gamma} + \sigma^2_{\beta\gamma} + \sigma^2_{\alpha\beta\gamma} + \sigma^2$

- **Expected Mean Squares.** ANOVA Model II - Random Factor Effects

Mean Square	df	Expected Mean Square
MSA	$a-1$	$\sigma^2 + nbc\frac{\sum \alpha_i^2}{a-1} + nc\sigma^2_{\alpha\beta} + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$
MSB	$b-1$	$\sigma^2 + nac\sigma^2_\beta + na\sigma^2_{\beta\gamma}$
MSC	$c-1$	$\sigma^2 + nab\sigma^2_\gamma + na\sigma^2_{\beta\gamma}$
$MSAB$	$(a-1)(b-1)$	$\sigma^2 + nc\sigma^2_{\alpha\beta} + n\sigma^2_{\alpha\beta\gamma}$
$MSAC$	$(a-1)(c-1)$	$\sigma^2 + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$
$MSBC$	$(b-1)(c-1)$	$\sigma^2 + na\sigma^2_{\beta\gamma}$
$MSABC$	$(a-1)(b-1)(c-1)$	$\sigma^2 + n\sigma^2_{\alpha\beta\gamma}$
MSE	$(n-1)abc$	σ^2

- **Example.** The table below contains the analysis of variance for a study of the effects of operators, machines, and batches on the daily output of a highly automated process. Each factor is assumed to be a random factor.

ANOVA Table for Balanced Three-Factor Study

Source of Variation	DF	Sum of Squares	Mean Square	F
Factor A (operators)	2	17.3		
Factor B (machines)	1	4.2		
Factor C (batches)	4	24.8		
AB Interactions		4.8		
AC Interactions		31.7		
BC Interactions		12.5		
ABC Interactions		11.9		
Error		137.7		
Total	89	244.9		

1. Test whether there is a significant interaction effect between operators and batches. Use $\alpha = 0.05$.

2. Test whether operators (factor A) have a main effect on output. Use $\alpha = 0.05$.

3. Use the *Satterwaite Approximate* F *Test* to test whether operators (factor A) have a main effect on output. Use $\alpha = 0.05$.
[Note: $E(MSA) = E(MSAB) + E(MSAC) - E(MSABC)$ when $\sigma^2_\alpha = 0$.]