Nested Designs

- **Training School Example**. A large manufacturing company operates three regional training schools for mechanics, one in each of its operating districts. The schools have two instructors each, who teach classes of about 15 mechanics in three-week sessions. The company was concerned about the effect of school (factor A) and instructor (factor B) on the learning achieved. To investigate these effects, classes in each district were formed in the usual way and then randomly assigned to one of the the two instructors in the school. This was done for two sessions, and at the end of each session a suitable summary measure of learning for the class was obtained. The results are shown below:

School	Instructor 1	Instructor 2
Atlanta	25	14
	29	11
Chicago	11	22
	6	18
San Francisco	17	5
	20	2

- **Nested Design Model**. Let Y_{ijk} denote the response for the kth trial when factor A is at the ith level and factor B is at the jth level. Both factors are assumed to be fixed.

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_{j(i)} + \epsilon_{ijk} \tag{1}$$

where:

1. $\mu_{..} = \frac{\sum_i \sum_j \mu_{ij}}{ab} = \frac{\sum_i \mu_{i.}}{a}$ is a constant with $\mu_{i.} = \frac{\sum_j \mu_{ij}}{b}$

2. $\alpha_i = \mu_{i.} - \mu_{..}$ are constants subject to the restriction $\sum_i \alpha_i = 0$

3. $\beta_{j(i)} = \mu_{ij} - \mu_{i.} = \mu_{ij} - \alpha_i - \mu_{..}$ are constants subject to the restriction $\sum_j \beta_{j(i)} = 0$ for all i

4. ϵ_{ijk} are independent $N(0, \sigma^2)$

5. $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$ (Equal sample sizes)

- **Estimation**. The least squares and maximum likelihood estimators of the parameters in nested design are given below:

Parameter	Estimator
$\mu_{..}$	$\hat{\mu}_{..} = \bar{Y}_{...}$
α_i	$\hat{\alpha}_i = \bar{Y}_{i..} - \bar{Y}_{...}$
$\beta_{j(i)}$	$\hat{\beta}_{j(i)} = \bar{Y}_{ij.} - \bar{Y}_{i..}$

- **Nested Design Model with Random Effects**. Both factors are assumed to be random.

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_{j(i)} + \epsilon_{ijk}$$

where:

1. $\mu_{..}$ is a constant

2. $\alpha_i, \beta_{j(i)}$, and ϵ_{ijk} are independent normal random variables with expectations 0 and variances σ_α^2, σ_β^2, and σ^2.

3. $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$ (Equal sample sizes)

- **Nested Design Model with Mixed Effects**. Factor A is fixed and B is random.

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_{j(i)} + \epsilon_{ijk}$$

where:

1. $\mu_{..}$ is a constant

2. α_i are constants subject to the restriction $\sum_i \alpha_i = 0$

3. $\beta_{j(i)}$ and ϵ_{ijk} are independent normal random variables with expectations 0 and variances σ^2_β, and σ^2.

4. $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$ (Equal sample sizes)

- **ANOVA Table for Nested Balanced Two-Factor Fixed Effects Model (B nested with A)**

Source of Variation	DF	Sum of Squares	Mean Square	E(MS)
Factor A	$a-1$	$SSA = bn\sum(\bar{Y}_{i..} - \bar{Y}_{...})^2$	$MSA = \frac{SSA}{a-1}$	$\sigma^2 + bn\frac{\sum_i \alpha_i^2}{a-1}$
Factor B (within A)	$a(b-1)$	$SSB(A) = n\sum\sum(\bar{Y}_{ij.} - \bar{Y}_{i..})^2$	$MSB(A) = \frac{SSB(A)}{a(b-1)}$	$\sigma^2 + n\frac{\sum\sum \beta_{j(i)}}{a(b-1)}$
Error	$ab(n-1)$	$SSE = \sum\sum\sum(\bar{Y}_{ijk} - \bar{Y}_{ij.})^2$	$MSE = \frac{SSE}{ab(n-1)}$	σ^2
Total	$abn-1$	$SSTO = \sum\sum\sum(\bar{Y}_{ijk} - \bar{Y}_{...})^2$		

- **R commands:**

```
response=c(25,29,14,11,11,6,22,18,17,20,5,2)
school=gl(3,4)
instructor=gl(2,2,12)

result=aov(response~school/instructor)
anova(result)
Response: response
                  Df  Sum Sq Mean Sq F value   Pr(>F)
school             2  156.5   78.25  11.179 0.009473 **
school:instructor  3  567.5  189.17  27.024 0.000697 ***
Residuals          6   42.0    7.00
```

- **Expected Mean Squares**. Nested Balanced Two-Factor Designs with Random Effects

Mean Square	Fixed A and Random B	Both Random
MSA	$\sigma^2 + bn\frac{\sum_i \alpha_i^2}{a-1} + n\sigma^2_\beta$	$\sigma^2 + bn\sigma^2_\alpha + n\sigma^2_\beta$
$MSB(A)$	$\sigma^2 + n\sigma^2_\beta$	$\sigma^2 + n\sigma^2_\beta$
MSE	σ^2	σ^2

- **Estimation of** $\mu_{i.}$: For Fixed Factor A.

1. $\hat{\mu}_{i.} = \bar{Y}_{i..}$

2. $s^2\{\bar{Y}_{i..}\} = \frac{MSE}{bn}$ with $df = ab(n-1)$ when B is also fixed.

3. $s^2\{\bar{Y}_{i..}\} = \frac{MSB(A)}{bn}$ with $df = a(b-1)$ when B is random.

4. The $(1-\alpha)100\%$ confidence interval: $\bar{Y}_{i..} \pm t_{[1-\alpha/2;df]} s\{\bar{Y}_{i..}\}$

5. *Example.* In the Training School example, suppose it was desired to estimate the mean learning score for the Atlanta school with 95% confidence interval.