Independence and More Expectations

- **Definition 5.8** Let Y_1 have the distribution function $F_1(y_1)$, Y_2 have the distribution function $F_2(y_2)$, and Y_1 and Y_2 have joint distribution function $F(y_1, y_2)$. Then Y_1 and Y_2 are said to be *independent* if and only if

$$F(y_1, y_2) = F_1(y_1)F_2(y_2)$$

  for every pair of real numbers (y_1, y_2).

  If Y_1 and Y_2 are not independent, they are said to be *dependent.*

- **Theorem 5.4** If Y_1 and Y_2 are two discrete random variables with joint probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y_1)$ and $p_2(y_2)$, respectively, then Y_1 and Y_2 are independent if and only if

$$p(y_1, y_2) = p_1(y_1)p_2(y_2)$$

  for all pairs of real numbers (y_1, y_2).

  If Y_1 and Y_2 are two continuous random variables with joint density function $F(y_1, y_2)$ and marginal probability functions $f_1(y_1)$ and $f_2(y_2)$, respectively, then Y_1 and Y_2 are independent if and only if

$$f(y_1, y_2) = f_1(y_1)f_2(y_2)$$

  for all pairs of real numbers (y_1, y_2).

- **Theorem 5.5** Let Y_1 and Y_2 have a joint density $f(y_1, y_2)$ that is positive if and only if $a \leq y_1 \leq b$ and $c \leq y_2 \leq d$, for constants a, b, c, and d; and $f(y_1, y_2) = 0$ otherwise. Then Y_1 and Y_2 are independent random variables if and only if

$$f(y_1, y_2) = g(y_1)h(y_2)$$

  where $g(y_1)$ is a nonnegative function of y_1 alone and $h(y_2)$ is a nonnegative function of y_2 alone.

- **Definition 5.9** Let $g(Y_1, Y_2, \ldots, Y_k)$ be a function of the discrete random variables, $Y_1, Y_2, \ldots, Y_k$, which have probability function $p(y_1, y_2, \ldots, y_k)$. Then the *expected value* of $g(Y_1, Y_2, \ldots, Y_k)$ is

$$E[g(Y_1, Y_2, \ldots, Y_k)] = \sum_{y_k} \cdots \sum_{y_2} \sum_{y_1} g(y_1, y_2, \ldots, y_k)p(y_1, y_2, \ldots, y_k)$$

  If $Y_1, Y_2, \ldots, Y_k$ are continuous random variables with joint density function $f(y_1, y_2, \ldots, y_k)$, then

$$E[g(Y_1, Y_2, \ldots, Y_k)] = \int_{y_k} \cdots \int_{y_2} \int_{y_1} g(y_1, y_2, \ldots, y_k) f(y_1, y_2, \ldots, y_k)\, dy_1\, dy_2 \ldots dy_k.$$

- **Special Theorems:**

  1. **Theorem 5.6.** Let c be a constant. Then $E(c) = c$.
  2. **Theorem 5.7.** Let $g(Y_1, Y_2)$ be a function of the random variables Y_1, Y_2, and let c be a constant. Then

$$E[cg(Y_1, Y_2)] = cE[g(Y_1, Y_2)].$$

  3. **Theorem 5.8.** Let Y_1 and Y_2 be random variables and $g_1(Y_1, Y_2), g_2(Y_1, Y_2), \ldots, g_k(Y_1, Y_2)$ be functions of Y_1 and Y_2. Then

$$E[g_1(Y_1, Y_2) + g_2(Y_1, Y_2) + \cdots + g_k(Y_1, Y_2)] = E[g_1(Y_1, Y_2)] + E[g_2(Y_1, Y_2)] + \cdots + E[g_k(Y_1, Y_2)].$$

  4. **Theorem 5.9.** Let Y_1 and Y_2 be independent random variables and $g(Y_1)$ and $h(Y_2)$ be functions of only Y_1 and Y_2, respectively, Then

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)]$$

     provided that the expectations exist.

- **Recommended problems:**
  Section 5.4: (pp. 251-255) # 45, 47, 49, 51, 53, 57, 61, 69, 71;
  Section 5.5/5.6: (pp. 261-264) # 73, 75, 77, 79, 81, 87.