Covariance and Linear Functions of R.V.

- **Definition 5.10.** If Y_1 and Y_2 are two random variables with means μ_1 and μ_2, respectively, the *covariance* of Y_1 and Y_2 is given by

$$Cov(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)].$$

The *correlation coefficient*, ρ, is defined as $\rho = \dfrac{Cov(Y_1, Y_2)}{\sigma_1 \sigma_2}$.

- **Theorem 5.10.** Let Y_1 and Y_2 be random variables with means μ_1 and μ_2, respectively, then

$$Cov(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = E(Y_1 Y_2) - E(Y_1)E(Y_2).$$

- **Theorem 5.11.** If Y_1 and Y_2 are independent random variables, then

$$Cov(Y_1, Y_2) = 0.$$

- **Theorem 5.12.** If $Y_1, Y_2, \ldots, Y_n$ and $X_1, X_2, \ldots, X_m$ be random variables, with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^{n} a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^{m} b_j X_j$$

for constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$. Then the following hold:

  1. $E(U_1) = \sum_{i=1}^{n} a_i \mu_i$.

  2. $V(U_1) = \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum\sum_{i<j} a_i a_j Cov(Y_i, Y_j)$, where the double sum is over all pairs (i, j) with $i < j$.

  3. $Cov(U_1, U_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(Y_i, X_j)$.

- **Recommended problems:**
Section 5.7: (pp. 268-270) # 89, 91, 93, 97.
Section 5.8: (pp. 276-279) # 103, 105, 107, 112.