Functions of Random Variables

- Three methods for finding the probability distribution of $U = h(Y_1, Y_2, \ldots, Y_n)$.

  1. **Distribution Function Method**
     - Find the region $U = u$ in the $(y_1, y_2, \ldots, y_n)$ space.
     - Find the region $U \leq u$.
     - Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, \ldots, y_n)$ over the region $U \leq u$.
     - Find the density function $f_U(u)$ by differentiating $F_U(u)$.
  2. **Transformation Method**. Let $U = h(Y)$, where $h(y)$ is either an increasing or decreasing function of y for all y such that $f_Y(y) > 0$.
     - Find the inverse function, $y = h^{-1}(u)$.
     - $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{dh^{-1}}{du} \right|$.
  3. **Method of Moment-Generating Functions**.
     - **Theorem 6.1** Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.
     - **Theorem 6.2** Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with moment-generating functions $m_{Y_1}(t), m_{Y_2}(t), \ldots, m_{Y_n}(t)$, respectively. If $U = Y_1 + Y_2 + \cdots + Y_n$, then $m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t)$.
     - **Theorem 6.3** Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed random variables with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$ and let $a_1, a_2, \ldots, a_n$ be constants. If $U = \displaystyle\sum_{i=1}^{n} a_i Y_i$, then U is normally distributed with $E(U) = \displaystyle\sum_{i=1}^{n} a_i \mu_i$ and $V(U) = \displaystyle\sum_{i=1}^{n} a_i^2 \sigma_i^2$.
     - **Theorem 6.4** Let $Y_1, Y_2, \ldots, Y_n$ be defined as in Theorem 6.3 and let $Z_i = \dfrac{Y_i - \mu_i}{\sigma_i}$, for $i = 1, 2, \ldots, n$. Then $\displaystyle\sum_{i=1}^{n} Z_i^2$ has a χ^2 distribution with n degrees of freedom.

- **Bivariate Transformation Method**. Suppose that Y_1 and Y_2 are continuous random variables with joint density function $f_{Y_1,Y_2}(y_1, y_2)$ and that for all (y_1, y_2) such that $f_{Y_1,Y_2}(y_1, y_2) > 0$

$$u_1 = h_1(y_1, y_2) \text{ and } u_2 = h_2(y_1, y_2)$$

is a one-to-one transformation from (y_1, y_2) to (u_1, u_2) with inverse

$$y_1 = h_1^{-1}(u_1, u_2) \text{ and } y_2 = h_2^{-1}(u_1, u_2).$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and *Jacobian*

$$J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1} \frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_2^{-1}}{\partial u_1} \frac{\partial h_1^{-1}}{\partial u_2} \neq 0$$

then the joint density of U_1 and U_2 is

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}\left(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)\right) |J|,$$

where $|J|$ is the absolute value of the Jacobian.