Sampling Distributions and the Central Limit Theorem

- **Definition 7.1** A *statistics* is a function of the observable random variables in a sample and known constants.

- **Theorem 7.1** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Then the sample mean, $\bar{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$, is normally distributed with mean $\mu_{\bar{Y}} = \mu$ and $\sigma^2_{\bar{Y}} = \sigma^2/n$.

- **Theorem 7.2** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Then $\sum_{i=1}^{n} Z_i = (Y_i - \mu)/\sigma$ are independent, standard normal random variables, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} Z_i^2$ has a χ^2 distribution with n degrees of freedom.

- **Theorem 7.3** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Then
$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$
has a χ^2 distribution with $(n-1)$ degrees of freedom. Also, $\bar{Y}$ and S^2 are independent random variables.

- **Definition 7.2** Let Z be a standard normal random variable and let W be a χ^2-distributed random variable with ν degrees of freedom. Then, if Z and W are independent, $T = \dfrac{Z}{\sqrt{W/\nu}}$ is said to be a t *distribution* with ν degrees of freedom.

- **Definition 7.3** Let W_1 and W_2 be independent χ^2-distributed random variables with ν_1 and ν_2 degrees of freedom, respectively. Then, $F = \dfrac{W_1/\nu_1}{W_2/\nu_2}$ is said to be an F *distribution* with ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom.

- **Theorem 7.4: Central Limit Theorem.** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a distribution with mean μ and finite variance σ^2. Then the distribution of the sample mean, $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, converges to the normal distribution with mean $\mu_{\bar{Y}} = \mu$ and $\sigma^2_{\bar{Y}} = \sigma^2/n$.

- **Examples:**

  1. A soft-drink machine is regulated so that it discharges an average of 200 ml per cup. If the amount of drink discharged is normally distributed with a standard deviation equal to 10 ml, what is the probability that a cup will contain less than 188 ml?

  2. In the previous problem, suppose we sample 25 cups from this machine and take their average content. What is the probability that, the average content of these 25 cups is less than 188 ml?

  3. The mean time it takes a senior high school student to complete a certain achievement test is 46.2 minutes with standard deviation of 8 minutes. If a random sample of 100 senior high school students who took the test was selected, find the probability that the average time it takes the group to complete the test will be

  a. less than 44 minutes.

  b. more than 48 minutes.

  c. between 44 to 48 minutes.