Random Variables and Probability Distributions

- **Definition.** For a given sample space S of some experiment, a *random variable* (r.v.) is any rule that associates a number with each outcome in S. In mathematical language, a *random variable* X is a function whose domain is the sample space and whose range is the set of real numbers.

  Example 1. Consider the experiment of tossing a fair coin 3 times.

  1. Let X be the number of heads.

  2. Let Y be the number of tails.

  Example 2. Consider the experiment of continually rolling a die until a condition is met.

  1. Let X be the number of trials until the first "6" results.

  2. Let Y be the number of trials until the third "6" results.

  Example 3. Consider a survey of 100 randomly chosen people,

  1. Let X be the gender of a person.

  2. Let Y be the height of a person.

  3. Let Z be the annual income of a person.

- **Two Types of Random Variables**.

  1. A random variable Y is said to be *discrete* if it can assume only a finite or countably infinite number of distinct values. (*There is a gap between possible values*).

  2. A random variable is said to be *continuous* if it takes on any value in an interval.

- **Bernoulli Trial.** Any experiment that has only two possible outcomes is called a *Bernoulli trial.*

  Examples

  1. Flipping a coin.

  2. Checking whether an item is defective or not.

  3. Correctly answering a true or false question.

  4. Passing an exam or not.

- **Bernoulli r.v.** Any random variable whose only possible values are 0 and 1 is called a *Bernoulli random varibale.*

- **Probability Distributions.** The *probability distribution* or *probability mass function* (pmf) of a discrete r.v. is defined for every number x by $p(x) = P(X = x) = P(all\ s \in S : X(s) = x)$.

  Example 4. Consider the experiment of rolling a pair of balanced dice. Let X represent the total number of dots on the top faces of the two dice. Construct the probability distribution of X.

Example 5. Consider the experiment of tossing a fair coin 3 times. Let X be the number of heads. Construct the probability distribution of X.

Example 6. Consider the experiment of tossing a loaded coin 3 times. If the probability of getting a head is 70%, construct the probability distribution table for the r.v. X that represents the number of heads. Construct the probability distribution of X.

- **Parameter.** Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a *parameter* of the distribution. The collection of all probability distributions for different values of the parameter is called a *family* of probability distributions.

- **Cumulative Distribution Function**. The *cumulative distribution function* (cdf) $F(x)$ of a discrete r.v. X with pmf $p(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y:y \leq x} p(y)$$

For any number x, $F(x)$ is the probability that the observed value of X will be at most x.

Example 7. Construct the cdf of the random variable described in example 6.

Expected Value of R.V.

- **Expected Value.** Let X be a discrete r.v. with set of possible values D and pmf $p(x)$. The *expected value* or *mean value* of X, denoted by $E(X)$ or μ_X, is

$$E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)$$

Example 8. Let X be the number of cars sold by Mike on a typical Saturday. Based on past experience, the probability of selling x cars is given below

x	0	1	2	3	4
$P(X = x)$	.20	.40	.25	0.10	0.05

Determine the expected number of cars the Mike will be able to sell this coming Saturday.

Example 9. Determine the expected value of a Bernoulli random variable X, if the probability of success is p.

Example 10. Consider the experiment of continually rolling a fair die until the first "6" results. Let X denote the number of trials needed to get this first "6".

1. Determine the probability distribution for X.

2. Show that the sum of the probabilities for all possible values of X is 1.

3. Determine the expected value of this random variable.

- **The Expected Value of a Function of R.V.** [Theorem 3.2] Let X be a discrete r.v. with set of possible values D and pmf $p(x)$. and be a X. Then for any real-valued function $g(X)$, the expected value of $g(X)$, denoted by $E[g(X)]$ or $\mu_{g(X)}$, is given by

$$E[g(X)] = \sum_{x \in D} g(x) \cdot p(x)$$

Proof:

Example 11. In example 2, if Mike earns a fix salary of \$50 per day and gets a commission of \$250 per car sold, determine his expected income on a typical Saturday.

- **Theorem.** $E(aX + b) = a \cdot E(X) + b$

  proof:

- **Theorem 3.5.** Let X be a discrete random variable with probability function $p(x)$ and $g_1(x), g_2(x), \ldots, g_k(x)$ be k functions of X. Then

$$E[g_1(x) + g_2(x) + \ldots + g_k(x)] = E[g_1(x)] + E[g_2(x)] + \ldots + E[g_k(x)]$$

- **Variance of R.V.** Let X have pmf $p(x)$ and expected value μ. Then the *variance* of X, denoted by $V(X)$ or σ_X^2, is

$$V(X) = \sum_{x \in D} (x - \mu)^2 \cdot p(x) = E[(X - \mu)^2].$$

- **Standard Deviation.** The *standard deviation* (SD) of X is $\sigma_X = \sqrt{\sigma_X^2}$.

  Example 12. Determine the standard deviation of X in example 8.

- **Theorem 3.6** Let X be a discrete r.v. with pmf $p(y)$; then

$$V(X) = E(X^2) - [E(X)]^2.$$

- Show that $V(aX + b) = a^2 V(X)$.

- **Recommended problems:**
  Section 3.2: (pp. 90-91) # 1, 5, 9.
  Section 3.3: (pp. 97-100) # 13, 15, 19, 23, 27, 31.