Chapter 4 - Continuous R.V. and Their Probability Distributions

- **Definition 4.1: (Cumulative) Probability Distribution.** Let Y denote any random variable. The *(cumulative) distribution function* of Y, denoted by $F(y)$, is given by $F(y) = P(Y \leq y)$ for $-\infty < y < \infty$
- **Theorem 4.1: Properties of a Distribution Function.** If $F(y)$ is a distribution function, then
  1. $F(-\infty) \equiv \lim_{y \to -\infty} F(y) = 0$
  2. $F(\infty) \equiv \lim_{y \to \infty} F(y) = 1$
  3. $F(y)$ is a nondecreasing function of y. (If y_1 and y_2 are any values such that $y_1 < y_2$, then $F(y_1) \leq F(y_2)$.)
- **Definition 4.2: Continuous R.V.** Let Y denote a random variable with distribution function $F(y)$, Y is said to be *continuous* if the distribution function $F(y)$ is continuous, for $-\infty < y < \infty$.

  Familiar Definition. A random variable Y is said to be *continuous* if its set of possible values is an entire interval of numbers – that is, if for some $A < B$, any number y between A and B is possible.
- **Definition 4.3: Probability Density Function.** Let $F(y)$ be a distribution function for a continuous random variable Y. Then $f(y)$, given by

  $$f(y) = \frac{dF(y)}{dy} = F'(y)$$

  wherever the derivative exists, is called the *probability density function* for the r.v. Y.
- **Theorem 4.2: Properties of a Density Function.** If $f(y)$ is a density function, then
  1. $f(y) \geq 0$ for any value of y.
  2. $\int_{-\infty}^{\infty} f(y)\, dy = 1$.
- **Definition 4.4**. Let Y denote any random variable. If $0 < p < 1$, the pth *quantile* of Y, denoted by ϕ_p, is the smallest value such that $P(Y \leq \phi_q) = F(\phi_p) \geq p$. If Y is continuous, ϕ_p is the smallest value such that $F(\phi_p) = P(Y \leq \phi_p) = p$. Some prefer to call ϕ_p the 100pth *percentile* of Y.
- **Theorem 4.3.** If the r.v. Y has density function $f(y)$ and $a \leq b$, then the probability that Y falls in the interval $[a, b]$ is

  $$P(a \leq Y \leq b) = \int_a^b f(y)\, dy$$

  That is, the probability that Y takes on a value in the interval $[a, b]$ is the area above this interval and under the graph of the density function.

  Hence, the *distribution function* $F(y)$ of a continuous r.v. Y with density function $f(y)$ is defined for every number y by

  $$F(y) = P(Y \leq y) = \int_{-\infty}^{y} f(y)\, dy$$

  For each y, $F(y)$ is the area under the density curve to the left of y.

- **Definition 4.5: Mean or Expected Value.** The expected value of a continuous r.v. Y with pdf $f(y)$ is

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy$$

provided that the integral exists.

- **Theorem 4.4: Expected Value of $g(Y)$.** Let $g(Y)$ be a function of Y; then the expected value of $g(Y)$ is given by

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f(y)\, dy$$

- **Theorem 4.5: Important Results.** Let c be a constant, and let $g(Y), g_1(Y), g_1(Y), \ldots, g_k(Y)$ be functions of a continuous r.v. Y. Then the following results hold:

  1. $E(c) = c$.

  2. $E[cg(Y)] = cE[g(Y)]$.

  3. $E[g_1(Y) + g_2(Y) + \cdots + g_k(Y)] = E[g_1(Y)] + E[g_2(Y)] + \cdots + E[g_k(Y)]$

- **Recommended problems:**
Section 4.2: (pp. 166-169) # 11, 15, 17, 19.
Section 4.3: (pp. 172-174) # 21, 27, 31, 33.