Joint Distributions

- **Definition 5.1: Joint Probability Function.** Let Y_1 and Y_2 be two discrete random variables. The *joint (or bivariate) probability function* for Y_1 and Y_2 is given by

$$p(y_1, y_2) = Pr(Y_1 = y_1, Y_2 = y_2), \qquad -\infty < y_1, y_2 < \infty.$$

The function $p(y_1, y_2)$ will be referred to as the *joint probability function* (jpf).

- **Theorem 5.1**. If Y_1 and Y_2 are two discrete random variables with joint probability function $p(y_1, y_2)$, then

  1. $p(y_1, y_2) \geq 0$ for all y_1, y_2.

  2. $\sum_{(y_1,y_2)} p(y_1, y_2) = 1$, where the sum is over all values (y_1, y_2) that are assigned nonzero probabilities.

- **Definition 5.2: Joint Distribution Function.** For any random variables Y_1 and Y_2, the *joint (or bivariate) distribution function* $F(y_1, y_2)$ is given by

$$F(y_1, y_2) = Pr(Y_1 \leq y_1, Y_2 \leq y_2), \qquad -\infty < y_1, y_2 < \infty.$$

- **Definition 5.3: Joint Probability Density Function.** Let Y_1 and Y_2 be two continuous random variables with joint distribution function $F(y_1, y_2)$. If there exists a nonnegative function $f(y_1, y_2)$ such that

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2) dt_2 dt_1$$

for all $-\infty < y_1, y_2 < \infty$, then Y_1 and Y_2 are said to *jointly continuous random variables.* The function $f(y_1, y_2)$ is called the *joint probability density function* (jpdf).

- **Theorem 5.2**. If Y_1 and Y_2 are two continuous random variables with joint distribution function $F(y_1, y_2)$, then

  1. $F(-\infty, -\infty) = F(-\infty, y_2) = F(y_1, -\infty) = 0$.

  2. $F(\infty, \infty) = 1$.

  3. if $y_1^* \geq y_1$ and $y_2^* \geq y_2$, then $F(y_1^*, y_2^*) - F(y_1^*, y_2) - F(y_1, y_2^*) + F(y_1, y_2) \geq 0$.

- **Theorem 5.3**. If Y_1 and Y_2 are two continuous random variables with joint density function $f(y_1, y_2)$, then

  1. $f(y_1, y_2) \geq 0$ for all y_1, y_2.

  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 dy_1 = 1$.

- **Recommended problems:**
  Section 5.2: (pp. 232-235) # 4, 5, 7, 9, 11, 13, 15, 17.

Marginal and Conditional Probability Distributions

- **Definition 5.4: Marginal Probability Distribution.**
  - Let Y_1 and Y_2 be jointly discrete random variables with probability function $p(y_1, y_2)$. Then the *marginal probability functions* of Y_1 and Y_2, respectively, are given by
  $$p_1(y_1) = \sum_{y_2} p(y_1, y_2) \quad \text{and} \quad p_2(y_2) = \sum_{y_1} p(y_1, y_2)$$
  - Let Y_1 and Y_2 be jointly continuous random variables with joint density function $f(y_1, y_2)$. Then the *marginal density functions* of Y_1 and Y_2, respectively, are given by
  $$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 \quad \text{and} \quad f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1.$$

- **Definition 5.5: Conditional Discrete Probability Function.** If Y_1 and Y_2 are jointly discrete random variables with probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y_1)$ and $p_2(y_2)$, respectively, then the *conditional discrete probability function* of Y_1 given Y_2 is
$$p(y_1|y_2) = P(Y_1 = y1|Y_2 = y_2) = \frac{P(Y_1 = y1, Y_2 = y_2)}{P(Y_2 = y_2} = \frac{p(y_1, y_2)}{p_2(y_2)}$$
provided that $p_2(y_2) > 0$.

- **Definition 5.6: Conditional Distribution Function (Continuous).** If Y_1 and Y_2 are jointly continuous random variables with joint density function $f(y_1, y_2)$ then the *conditional distribution function* of Y_1 given $Y_2 = y_2$ is
$$F(y_1|y_2) = P(Y_1 \leq y_1|Y_2 = y_2).$$

- **Definition 5.7: Conditional Density Function.** Let Y_1 and Y_2 be jointly continuous random variables with joint density function $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. For any y_2 such that $f_2(y_2) > 0$, the *conditional density function* of Y_1 given $Y_2 = y_2$ is given by
$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_{Y_2}(y_2)}$$
and, for any y_1 such that $f_1(y_1) > 0$, the *conditional density function* of Y_2 given $Y_1 = y_1$ is given by
$$f(y_2|y_1) = \frac{f(y_1, y_2)}{f_{Y_1}(y_1)}.$$

- **Recommended problems:**
Section 5.3: (pp. 242-247) # 19, 20, 21, 22, 23, 25, 27, 29, 31, 33, 37, 39.