STAT 441 - Mathematical Statistics I
Due: Wednesday (at the start of class), Oct. 11, 2017

Instructions: *Include all relevant work to get full credit.*

a) $F(x) = \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$

Homework 10

1. Suppose X has an exponential distribution with parameter $\beta = \frac{1}{\lambda}$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

$\Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$

   a. Find cumulative distribution function of X, $F(x)$.

   b. If $P(X > 2) = 0.0821$, find $\lambda, E(Y)$, and $P(X \leq 1.7)$.

b) $P(X>2) = 1 - F(2) = e^{-2\lambda} = .0821 \Rightarrow \lambda = -\frac{1}{2}\ln(.0821) \approx 1.25$

$\Rightarrow E(X) = \frac{1}{\lambda} = \frac{1}{1.25} = .8$ and $P(Y \leq 1.7) = 1 - e^{-1.25(1.7)} \approx .8806$

2. Suppose X has an exponential distribution with parameter $\beta = \frac{1}{\lambda}$, show that $P(X > a+b|X > a) = P(X > b)$ for any $a > 0$ and $b > 0$.

3. Suppose $Y \sim Gamma(\alpha, \beta)$.

   a. Prove that $V(Y) = \alpha\beta^2$. *You may use the fact that* $E(Y) = \alpha\beta$.

   b. If a is any positive or negative value such that $\alpha + a > 0$, show that $E(Y^a) = \dfrac{\beta^a \Gamma(\alpha + a)}{\Gamma(\alpha)}$.

   c. Why did your answer in part (b) require that $\alpha + a > 0$?

   d. Use the result in part (b) to give an expression for $E(1/Y)$.

#2 Note from part (1), $P(X > x) = 1 - F(x) = e^{-\lambda x}$

$$\Rightarrow P(X > a+b \mid X > a) = \frac{P(\{X > a+b\} \cap \{X > a\})}{P(X > a)} = \frac{P(X > a+b)}{P(X > a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b} = P(X > b) \blacksquare$$

#3 a) $Y \sim Gamma(\alpha, \beta) \Rightarrow f(y) = \dfrac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}, \quad y \geq 0$

$$\Rightarrow E(Y^2) = \int_0^\infty y^2 \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}\,dy = \frac{\beta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)} \underbrace{\int_0^\infty \frac{y^{(\alpha+2)-1} e^{-y/\beta}}{\beta^{\alpha+2}\Gamma(\alpha+2)}\,dy}_{=1} = \frac{\beta^2(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$\Rightarrow V(Y) = E(Y^2) - \mu_Y^2 = \beta^2(\alpha^2 + \alpha) - (\alpha\beta)^2 = \alpha\beta^2 \blacksquare$$

b) $$E(Y^a) = \int_0^\infty \frac{y^a y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)}\,dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty y^{(\alpha+a)-1} e^{-y/\beta}\,dy}_{= \beta^{\alpha+a}\Gamma(\alpha+a)}$$

$$= \frac{\beta^{\cancel{\alpha}+a}\Gamma(\alpha+a)}{\cancel{\beta^\alpha}\Gamma(\alpha)} = \frac{\beta^a \Gamma(\alpha+a)}{\Gamma(\alpha)} \blacksquare$$

c) For the Gamma function $\Gamma(t)$, we need $t > 0$.

d) $$E\left(\frac{1}{Y}\right) = E(Y^{-1}) = \frac{\beta^{-1}\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{\Gamma(\alpha-1)}{\beta(\alpha-1)\Gamma(\alpha-1)} = \frac{1}{\beta(\alpha-1)}.$$