STAT 441 - MATHEMATICAL STATISTICS I
DUE: FRIDAY (AT THE START OF CLASS), OCT. 13, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 11

1. Suppose that the waiting time for the first customer to enter a retail shop after 9:00 AM is a random variable X has an exponential distribution with parameter $\beta = \frac{1}{\lambda}$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

   a. Find the moment-generating function for X.

   b. Use the answer from part (a) to find $E(X)$ and $V(X)$.

2. If X is a random variable with moment-generating function $m_X(t)$ and Y is given by $Y = aX + b$, show that the moment-generating function of Y is $m_Y(t) = e^{tb} m_X(at)$.

3. If $Z \sim N(0,1)$, show that the moment-generating function of Z is $m_Z(t) = e^{\frac{1}{2}t^2} = exp(\frac{1}{2}t^2)$.
   [Hint: Use the fact that the total area under the density function of $N(\mu = t, \sigma = 1)$ is 1.]

4. If $Z \sim N(0,1)$ and $Y = \sigma Z + \mu$, Y will be normally distributed with mean μ and standard deviation σ. Use the results from parts (2) and (3) to show that the moment-generating function of a normal random variable Y with mean μ and standard deviation σ is $m_Y(t) = exp\left(\mu t + \dfrac{t^2\sigma^2}{2}\right)$

5. Use the uniqueness of moment-generating functions to determine the exact distribution (give the name and parameter values) of the random variables that have each of the following moment-generating functions:

   a. $m(t) = (1-4t)^{-2}$ $\Rightarrow$ Gamma$(\alpha = 2, \beta = 4)$

   b. $m(t) = 1/(1-3.2t)$ $\Rightarrow$ Exp$(\lambda = 1/3.2 = .3125) \equiv$ Gamma$(\alpha = 1, \beta = 3.2)$

   c. $m(t) = e^{-5t+6t^2}$ $\Rightarrow$ $N(\mu = -5, \sigma^2 = 12)$

Solutions:

1) (a) $m(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^\infty$

$\Rightarrow$ for $t < \lambda$, $m(t) = \frac{\lambda}{t-\lambda}(0 - e^0) = -\lambda(t-\lambda)^{-1} = \frac{\lambda}{\lambda - t}$

(b) $E(X) = m'(0) = \lambda(t-\lambda)^{-2}\Big|_{t=0} = \lambda(0-\lambda)^{-2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$.

$E(X^2) = m''(0) = -2\lambda(t-\lambda)^{-3}\Big|_{t=0} = \frac{-2\lambda}{-\lambda^3} = \frac{2}{\lambda^2} \Rightarrow V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

2) $m_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E(e^{tb}e^{atX}) = e^{tb}E(e^{atX}) = e^{tb} m_X(at)$ ■

3) $m_Z(t) = \int_{-\infty}^\infty e^{tz} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2t + t^2)} e^{\frac{1}{2}t^2} dz$

$= e^{\frac{1}{2}t^2} \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz}_{\text{density function of } N(\mu = t, \sigma = 1)} = e^{\frac{1}{2}t^2}$. ■

4) If $Y = \sigma Z + \mu$

$\Rightarrow m_Y(t) = e^{t\mu} m_Z(\sigma t) = e^{\mu t} \cdot e^{\frac{1}{2}(\sigma t)^2} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$. ■