STAT 441 - Mathematical Statistics I
Due: Monday (at the start of class), Oct. 16, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 12

1. Do all parts of problem 5.7 on page 233.
2. Do all parts of problem 5.9 on page 233.
3. Do all parts of problem 5.12 on page 234.
4. Do all parts of problem 5.14 on page 234.
5. Do all parts of problem 5.15 on page 235.

Solutions:

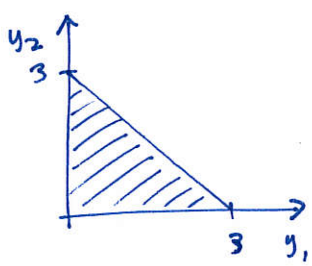
#1 $f(y_1,y_2) = \begin{cases} e^{-(y_1+y_2)} & , y_1>0, y_2>0 \\ 0 & , \text{elsewhere} \end{cases}$

a) $P(Y_1<1, Y_2>5) = \int_0^1\int_5^\infty (e^{-y_1}e^{-y_2}dy_2)\,dy_1 = \int_0^1 \left(-e^{-y_1}e^{-y_2}\Big|_{y_2=5}^{y_2=\infty}\right)dy_1$

$= \int_0^1 -e^{-y_1}(0-e^{-5})\,dy_1 = e^{-5}\int_0^1 e^{-y_1}dy_1$

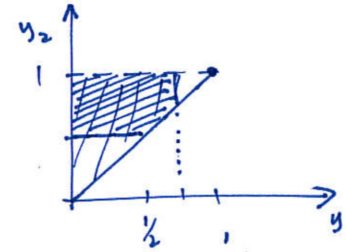
$= e^{-5}\left(-e^{-y_1}\Big|_0^1\right) = e^{-5}\left(-e^{-1}+e^0\right) = \frac{1}{e^5}\left(1-\frac{1}{e}\right) \approx .00426$

b) $P(Y_1+Y_2<3) = \int_0^3\int_0^{3-y_1} e^{-y_1}e^{-y_2}dy_2dy_1 = \int_0^3\left(-e^{-y_1}e^{-y_2}\Big|_{y_2=0}^{y_2=3-y_1}\right)dy_1$



$= \int_0^3 -e^{-y_1}(e^{y_1-3}-e^0)\,dy_1 = \int_0^3(-e^{-3}+e^{-y_1})\,dy_1$

$= -e^{-3}y_1\Big|_0^3 - e^{-y_1}\Big|_0^3 = -3e^{-3} - (e^{-3}-1) = 1-\frac{4}{e^3} \approx .8$



#2 $f(y_1,y_2) = \begin{cases} k(1-y_2) & , 0\le y_1\le y_2\le 1 \\ 0 & , \text{else where} \end{cases}$

a) $\int_0^1\int_{y_1}^1 k(1-y_2)\,dy_2dy_1 = 1$

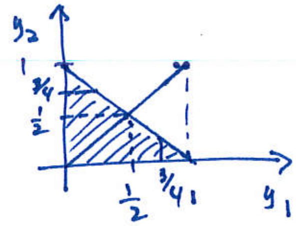
$\Rightarrow \int_0^1\left(k\left(y_2-\frac{y_2^2}{2}\right)\Big|_{y_1}^1\right)dy = \int_0^1 k\left(\frac{1}{2}-y_1+\frac{y_1^2}{2}\right)dy_1 = k\left(\frac{1}{2}y_1-\frac{1}{2}y_1^2+\frac{1}{6}y_1^3\right)\Big|_0^1 = 1$

$\Rightarrow k = 1/\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{1/6} = 6$

b) $P\left(Y_1\le\frac{3}{4}, Y_2\ge\frac{1}{2}\right) = \int_0^{1/2}\int_{1/2}^1 6(1-y_2)\,dy_2dy_1 + \int_{1/2}^{3/4}\int_{y_1}^1 6(1-y_2)\,dy_2dy_1$

$= \frac{3}{8} + \frac{7}{64} = \frac{31}{64}$

#3 $f(y_1, y_2) = \begin{cases} 2, & 0 \le y_1 \le 1, \ 0 \le y_2 \le 1, \ 0 \le y_1 + y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$



a) $P(Y_1 \le 3/4, Y_2 \le 3/4) = \int_0^{1/4}\int_0^{3/4} 2\, dy_2\, dy_1 + \int_{1/4}^{3/4}\int_0^{1-y_1} 2\, dy_2\, dy_1$

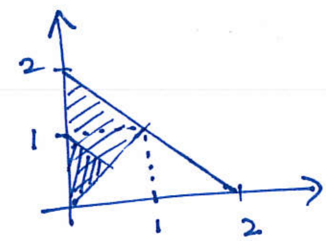
$= \int_0^{1/4} \left(2y_2 \Big|_0^{3/4}\right) dy_1 + \int_{1/4}^{3/4} \left(2y_2 \Big|_0^{1-y_1}\right) dy_1$

$= \int_0^{1/4} \frac{3}{2}\, dy_1 + \int_{1/4}^{3/4} 2(1-y_1)\, dy_1$

$= \frac{3}{2}\left(\frac{1}{4}\right) + 2\left(y_1 - \frac{1}{2}y_1^2\right)\Big|_{1/4}^{3/4} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$

b) $P(Y_1 \le \frac{1}{2}, Y_2 \le \frac{1}{2}) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 2\, dy_2\, dy_1 = \int_0^{\frac{1}{2}} 2\left(\frac{1}{2}\right) dy_1 = y_1 \Big|_0^{\frac{1}{2}} = \frac{1}{2}$.

#4 $f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2, \ y_1 + y_2 \le 2 \\ 0, & \text{elsewhere} \end{cases}$



a) $\int_0^1\int_{y_1}^{2-y_1} 6y_1^2 y_2\, dy_2\, dy_1 = \int_0^1 \left(6y_1^2 \frac{1}{2}y_2^2 \Big|_{y_1}^{2-y_1}\right) dy_1$

$= \int_0^1 3y_1^2\left((2-y_1)^2 - y_1^2\right) dy_1 = \int_0^1 3y_1^2(4-4y_1)\, dy_1$

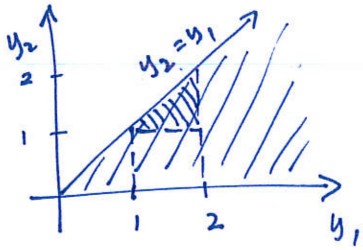
$= \int_0^1 (12y_1^2 - 12y_1^3)\, dy_1 = (4y_1^3 - 3y_1^4)\Big|_0^1 = 4 - 3 = 1$

b) $P(Y_1 + Y_2 < 1) = \int_0^{\frac{1}{2}}\int_{y_1}^{1-y_1} 6y_1^2 y_2\, dy_2\, dy_1 = \int_0^{\frac{1}{2}}\left(3y_1^2 y_2^2 \Big|_{y_1}^{1-y_1}\right) dy_1 = \int_0^{\frac{1}{2}} 3y_1^2\left((1-y_1)^2 - y_1^2\right) dy_1$

$= \int_0^{1/2} 3y_1^2(1-2y_1)\, dy_1 = \int_0^{1/2}(3y_1^2 - 6y_1^3)\, dy_1$

$= \left(y_1^3 - \frac{3}{2}y_1^4\right)\Big|_0^{1/2} = \left(\frac{1}{2}\right)^3 - \frac{3}{2}\left(\frac{1}{2}\right)^4 = \frac{1}{32}$

#5 $f(y_1, y_2) = \begin{cases} e^{-y_1} & , 0 \le y_2 \le y_1 < \infty \\ 0 & , \text{elsewhere} \end{cases}$



a) $P(Y_1 < 2, Y_2 > 1) = \int_1^2 \int_1^{y_1} e^{-y_1} dy_2 \, dy_1 = \int_1^2 \left(e^{-y_1} y_2 \Big|_1^{y_1} \right) dy_1$

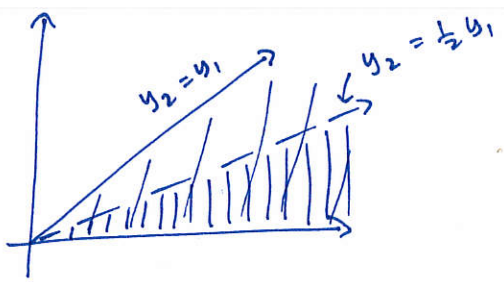
$= \int_1^2 (y_1 e^{-y_1} - e^{-y_1}) \, dy_1 = -y_1 e^{-y_1} \Big|_1^2 + \cancel{\int_1^2 e^{-y_1} dy_1} - \cancel{\int_1^2 e^{-y_1} dy_1}$

let $u = y_1 \quad dv = e^{-y_1} dy_1$

$du = dy_1 \quad v = -e^{-y_1} + c$

$= -2e^{-2} + e^{-1} \approx .0972$

b) $P(Y_1 \ge 2Y_2) = \int_0^\infty \int_0^{\frac{1}{2} y_1} e^{-y_1} dy_2 \, dy_1 = \int_0^\infty \left(e^{-y_1} y_2 \Big|_0^{\frac{1}{2} y_1} \right) dy_1$



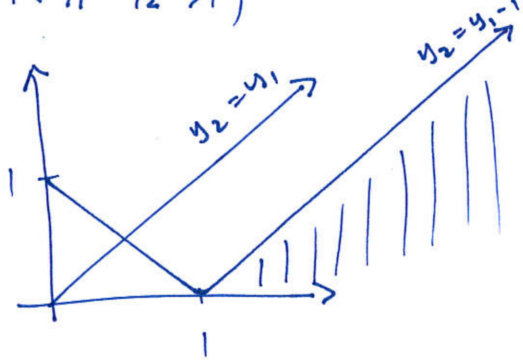
$= \int_0^\infty \frac{1}{2} y_1 e^{-y_1} dy_1$

$= \frac{1}{2} \left(-y_1 e^{-y_1} \Big|_0^\infty + \int_0^\infty e^{-y_1} dy_1 \right)$

$\overset{L'H}{=} \frac{1}{2}(0+0) + \frac{1}{2} \left(-e^{-y} \Big|_0^\infty \right) = \frac{1}{2}(0+1)$

$= \frac{1}{2}$

c) $P(Y_1 - Y_2 \ge 1) = P(Y_2 \le Y_1 - 1) = \int_1^\infty \int_0^{y_1 - 1} e^{-y_1} dy_2 \, dy_1$



$= \int_1^\infty e^{-y_1}(y_1 - 1) \, dy_1$

$= \int_1^\infty y_1 e^{-y_1} dy_1 - \int_1^\infty e^{-y_1} dy_1$

$= -y_1 e^{-y_1} \Big|_1^\infty + \cancel{\int_1^\infty e^{-y_1} dy_1} - \cancel{\int_1^\infty e^{-y_1} dy_1}$

$\overset{L'H}{=} 0 + e^{-1} = \frac{1}{e} \approx .368$