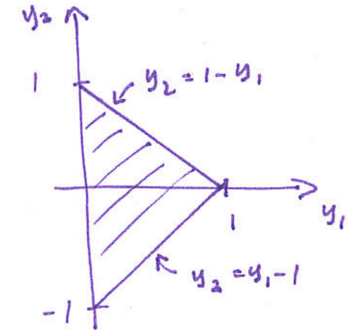
STAT 441 - MATHEMATICAL STATISTICS I
DUE: FRIDAY (AT THE START OF CLASS), OCT. 20, 2017

Instructions: *Include all relevant work to get full credit.*

Homework 13

1. Do all parts of problem 5.31 on page 245.
2. Do all parts of problem 5.77 on page 262.



Solutions:

#1 a) $f(y_1,y_2) = \begin{cases} 30y_1y_2^2, & y_1 - 1 \le y_2 \le 1 - y_1,\ 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$$\Rightarrow f_{Y_1}(y_1) = \int_{y_1-1}^{1-y_1} 30y_1y_2^2\,dy_2 = 30y_1\left(\frac{1}{3}y_2^3\Big|_{y_1-1}^{1-y_1}\right)$$

$$= 10y_1\left((1-y_1)^3 - (y_1-1)^3\right), \quad 0 \le y_1 \le 1$$

$$= 10y_1\left((1-y_1)^3 + (1-y_1)^3\right), \quad 0 \le y_1 \le 1$$

$$= 20y_1(1-y_1)^3, \quad 0 \le y_1 \le 1$$

$$= \frac{\Gamma(2+4)}{\Gamma(2)\Gamma(4)} y_1^{2-1}(1-y_1)^{4-1} \equiv \text{Beta}(\alpha=2, \beta=4)$$

b) $f_{Y_2}(y_2) = \begin{cases} \int_0^{y_2+1} 30y_1y_2^2\,dy_1, & -1 \le y_2 \le 0 \\ \int_0^{1-y_2} 30y_1y_2^2\,dy_1, & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

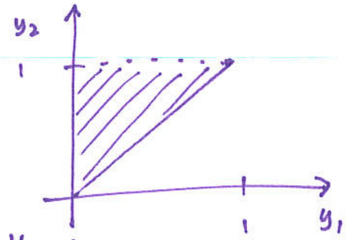
$$= \begin{cases} 15y_1^2y_2^2\Big|_{y_1=0}^{y_1=y_2+1} = 15y_2^2\left((y_2+1)^2 - 0\right) = 15y_2^2(y_2+1)^2, & -1 \le y_2 \le 0 \\ 15y_1^2y_2^2\Big|_{y_1=0}^{y_1=1-y_2} = 15y_2^2\left((1-y_2)^2 - 0\right) = 15y_2^2(1-y_2)^2, & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

c) $$f_{Y_2|Y_1}(y_2|y_1) = \frac{f(y_1,y_2)}{f_{Y_1}(y_1)} = \frac{30y_1y_2^2}{20y_1(1-y_1)^3} = \begin{cases} \frac{3y_2^2}{2(1-y_1)^3}, & y_1 - 1 \le y_2 \le 1 - y_1 \\ 0, & \text{elsewhere} \end{cases}$$

d) $$P(Y_2 > 0 \mid Y_1 = .75) = \int_0^{1-.75} \frac{3y_2^2}{2(1-.75)^3}\,dy_2 = \int_0^{.25} 96y_2^2\,dy_2 = 32y_2^3\Big|_0^{.25}$$

$$= 32(.25)^3 = \frac{1}{2}$$

#2 $f(y_1,y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$



a) $$E(Y_1) = \int_0^1\int_{y_1}^1 y_1\, 6(1-y_2)\, dy_2\, dy_1 = \int_0^1 6y_1\left(y_2 - \tfrac{1}{2}y_2^2\right)\Big|_{y_2=y_1}^{y_2=1} dy_1$$

$$= \int_0^1 6y_1\left(\tfrac{1}{2} - y_1 + \tfrac{1}{2}y_1^2\right) dy_1 = \int_0^1 (3y_1 - 6y_1^2 + 3y_1^3)\, dy_1$$

$$= \left(\tfrac{3}{2}y_1^2 - 2y_1^3 + \tfrac{3}{4}y_1^4\right)\Big|_0^1 = \tfrac{3}{2} - 2 + \tfrac{3}{4} = \frac{6-8+3}{4} = \frac{1}{4}$$

$$E(Y_2) = \int_0^1\int_{y_1}^1 y_2\, 6(1-y_2)\, dy_2\, dy_1 = \int_0^1 (3y_2^2 - 2y_2^3)\Big|_{y_1}^1 dy_1$$

$$= \int_0^1 (1 - 3y_1^2 + 2y_1^3)\, dy_1 = \left(y_1 - y_1^3 + \tfrac{1}{2}y_1^4\right)\Big|_0^1 = 1 - 1 + \tfrac{1}{2} = \tfrac{1}{2}$$

b) $$V(Y_1) = E(Y_1^2) - (E(Y_1))^2 = \frac{1}{10} - \left(\frac{1}{4}\right)^2 = \frac{1}{10} - \frac{1}{16} = \frac{6}{160} = \frac{3}{80} \approx .0375$$

$$E(Y_1^2) = \int_0^1\int_{y_1}^1 y_1^2\, 6(1-y_2)\, dy_2\, dy_1 = \int_0^1 6y_1^2\left(\tfrac{1}{2} - y_1 + \tfrac{1}{2}y_1^2\right) dy_1 = \int_0^1 (3y_1^2 - 6y_1^3 + 3y_1^4)\, dy_1$$

$$= \left(y_1^3 - \tfrac{6}{4}y_1^4 + \tfrac{3}{5}y_1^5\right)\Big|_0^1 = 1 - \tfrac{3}{2} + \tfrac{3}{5} = \frac{10-15+6}{10} = \frac{1}{10}$$

$$E(Y_2^2) = \int_0^1\int_{y_1}^1 y_2^2\, 6(1-y_2)\, dy_2\, dy_1 = \int_0^1 \left(2y_2^3 - \tfrac{6}{4}y_2^4\right)\Big|_{y_1}^1 dy_1$$

$$= \int_0^1 \left(\tfrac{1}{2} - 2y_1^3 + \tfrac{3}{2}y_1^4\right) dy_1 = \left(\tfrac{1}{2}y_1 - \tfrac{1}{2}y_1^4 + \tfrac{3}{10}y_1^5\right)\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{3}{10} = \tfrac{3}{10}$$

$$\Rightarrow V(Y_2) = E(Y_2^2) - (E(Y_2))^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{6-5}{20} = \frac{1}{20}$$

c) $$E(Y_1 - 3Y_2) = E(Y_1) - 3E(Y_2) \quad \text{by Theorems 5.7 and 5.8}$$

$$= \frac{1}{4} - 3\left(\frac{1}{2}\right) = -\frac{5}{4}$$